A CLOSED FORM SOLUTION TO $L_2$-SENSITIVITY MINIMIZATION OF SECOND-ORDER STATE-SPACE DIGITAL FILTERS SUBJECT TO $L_2$-SCALING CONSTRAINTS

Shunsuke Yamaki, Masahide Abe, and Masayuki Kawamata

Department of Electronic Engineering, Graduate School of Engineering, Tohoku University 6-6-05, Aoba, Aramaki, Aoba-ku, Sendai, 980-8579
Phone:+81-22-795-7095, Fax:+81-22-263-9169, Email:yamaki@mk.ecei.tohoku.ac.jp

ABSTRACT—This paper proposes a closed form solution to $L_2$-sensitivity minimization of 2nd-order state-space digital filters subject to $L_2$-scaling constraints. The proposed solution reduces a constrained optimization problem to an unconstrained optimization problem by appropriate variable transformation. Furthermore, by restricting ourselves to the case of 2nd-order state-space digital filters, we can formulate the $L_2$-sensitivity minimization problem via hyperbolic functions. As a result, we can express the $L_2$-sensitivity in closed form, whose minimization subject to $L_2$-scaling constraints is achieved without iterative calculations.

1. INTRODUCTION

$L_2$-sensitivity is one of the functions that evaluates the coefficient quantization effects of state-space digital filters [1–4]. It is difficult to minimize $L_2$-sensitivity by analytical method since the $L_2$-sensitivity minimization problem is a nonlinear problem. Previously, with respect to the $L_2$-sensitivity minimization problem, we proposed a closed form solution for the case of 2nd-order digital filters without $L_2$-scaling constraints [3].

The authors in [2] investigated the $L_2$-sensitivity minimization problem subject to $L_2$-scaling constraints. It is known that the scaling constraints are necessary for preventing overflow of state variables [5]. The authors in [2] proposed solutions to this problem, however, these solutions require many iterative calculations. We proposed a novel solution which achieves the $L_2$-sensitivity minimization with fast convergence [4].

In this paper, we newly propose a closed form solution to the $L_2$-sensitivity minimization problem subject to $L_2$-scaling constraints. We construct this solution by combining two solutions from [3] and [4], and derive the minimum $L_2$-sensitivity realization.

2. $L_2$-SENSITIVITY OF STATE-SPACE DIGITAL FILTERS

For a given $N$th-order transfer function $H(z)$, a state-space digital filter can be described as follows:

$$x(n+1) = Ax(n) + bu(n)$$

$$y(n) = cx(n) + du(n)$$

where $x(n)$ is an $N \times 1$ state-vector, $u(n)$ is a scalar input, $y(n)$ is a scalar output and $A$, $b$, $c$, $d$ are real constant matrices of appropriate dimensions. The transfer function of the digital filter $(A, b, c, d)$ is given by

$$H(z) = c(zI - A)^{-1}b + d.$$  (3)

The $L_2$-sensitivity of the filter $H(z)$ with respect to the realization $(A, b, c, d)$ is defined by

$$S(A, b, c) = \left\| \frac{\partial H(z)}{\partial A} \right\|_2^2 + \left\| \frac{\partial H(z)}{\partial b} \right\|_2^2 + \left\| \frac{\partial H(z)}{\partial c} \right\|_2^2$$

$$= \text{tr}(W_0)\text{tr}(K_0) + \text{tr}(W_0) + \text{tr}(K_0) +2 \sum_{i=1}^{\infty} \text{tr}(W_i)\text{tr}(K_i)$$

where $\| \cdot \|_2$ denotes the $L_2$-norm of function $(\cdot)$, and $K_i$ and $W_i$ are the general controllability and observability gramian of a filter $(A, b, c, d)$, respectively [1]. Under the coordinate transformation defined by $x(n) = T^{-1}x(n)$, we have the $L_2$-sensitivity of a transformed filter $(T^{-1}AT, T^{-1}b, cT, d)$ as follows:

$$S(P) = \text{tr}(W_0P)\text{tr}(K_0P^{-1})$$

$$+ \text{tr}(W_0P) + \text{tr}(K_0P^{-1})$$

$$+ 2 \sum_{i=1}^{\infty} \text{tr}(W_iP)\text{tr}(K_iP^{-1})$$

where $P$ is a positive definite symmetric matrix defined by $P = TT^T$ [1].

3. $L_2$-SENSITIVITY MINIMIZATION SUBJECT TO $L_2$-SCALING CONSTRAINTS

In this section, we discuss the $L_2$-sensitivity minimization considering constraints which prevent overflow. Our solution to the $L_2$-sensitivity minimization subject to $L_2$-scaling constraints is constructed by combining two solutions from [3] and [4], and derives the minimum $L_2$-sensitivity realization in closed form without iterative calculations.

3.1 Formulation of the Problem

We adopt a input normal realization as an initial realization to synthesize the $L_2$-sensitivity minimization...
problem. The input normal realization is one of the filter structures whose controllability and observability graminians satisfy the following conditions:

\[ K_0 = I, \ W_0 = \Theta^2, \ \Theta = \text{diag}(\theta_1, \theta_2, \ldots, \theta_N). \tag{6} \]

The parameters \( \theta_i (i = 1, 2, \ldots, N) \) are the second order modes of the filter \( H(z) \) [5].

In order to prevent overflow of state variables, the variance of state variables must be unity under the white gaussian input with zero mean and unit variance. Under the above constraints, \( L_2 \)-sensitivity minimization subject to \( L_2 \)-scaling constrains is formulated as follows:

\[
\begin{align*}
\text{minimize the } L_2 \text{-sensitivity } S(P) \text{ w.r.t. } P \\
\text{subject to } \text{tr}(P^{-1}) = N.
\end{align*}
\tag{7}
\]

### 3.2 Our Proposed Solution

In order to reduce the constraint in problem (7), we newly propose a variable transformation of positive definite symmetric matrix \( P \) as follows:

\[ P = \frac{\text{tr}(Q^{-1})}{N} Q \tag{8} \]

where \( Q \) is an arbitrary positive definite symmetric matrix. If a positive definite symmetric matrix \( P \) satisfies \( \text{tr}(P^{-1}) = N \), there surely exists a positive definite symmetric matrix \( Q \) which satisfies Eq. (8), and vice versa. Substituting Eqs. (8) and (6) into Eq. (5), we have a novel expression of \( L_2 \)-sensitivity \( \bar{S}(Q) \) with variable matrix \( Q \) such as

\[
\bar{S}(Q) = N + \frac{N}{N} \text{tr}(W_0 Q) \text{tr}(IQ^{-1}) + 2 \sum_{i=1}^{\infty} \text{tr}(W_i Q) \text{tr}(K_i Q^{-1}). \tag{9}
\]

The constrained optimization problem (7) is thus reduced to the unconstrained optimization problem as follows [4]:

\[
\begin{align*}
\text{minimize the } L_2 \text{-sensitivity } \bar{S}(Q) \text{ w.r.t. } Q \\
\text{where } Q \text{ is an arbitrary positive definite symmetric matrix.}
\end{align*}
\tag{10}
\]

We newly proved that the solution to the optimization problem (10), the positive definite symmetric matrix \( Q \) which minimizes \( \bar{S}(Q) \), is expressed as

\[
Q = \Theta^{-\frac{1}{2}} \begin{bmatrix} \cosh(q) & \sinh(q) \\ \sinh(q) & \cosh(q) \end{bmatrix} \Theta^{-\frac{1}{2}} (q \in R) \tag{11}
\]

without loss of generality. Substituting Eq. (11) into Eq. (9) yields the \( L_2 \)-sensitivity \( \bar{S}(Q) \) of 2nd-order digital filters expressed as

\[
\bar{S}(Q) = \bar{S}(q) = \sum_{n=1}^{1} \bar{c}_n e^{2qn} \tag{12}
\]

where \( \bar{c}_n (n = -1, 0, 1) \) are constants determined by the coefficient matrices. Solving \( \partial \bar{S}(q) / \partial q = 0 \) with respect to \( q \) derives the optimal solution \( q_{\text{opt}} \) as

\[
q_{\text{opt}} = \frac{1}{4} \log \left( \frac{\bar{c}_{-1}}{\bar{c}_1} \right). \tag{13}
\]

The minimum \( L_2 \)-sensitivity \( \bar{S}_{\text{min}} \) is expressed by substituting Eq. (13) into Eq. (12) as

\[
\bar{S}_{\text{min}} = \bar{S}(q_{\text{opt}}) = 2 \sqrt{\bar{c}_{-1} \bar{c}_1 + \bar{c}_0}. \tag{14}
\]

### 4. CONCLUSION

This paper has proposed a closed form solution to the \( L_2 \)-sensitivity minimization problem subject to \( L_2 \)-scaling constraints. Our proposed solution reduces a constrained optimization problem to an unconstrained optimization problem, and it describes the \( L_2 \)-sensitivity in closed form using hyperbolic functions. As a result, we can derive the minimum \( L_2 \)-sensitivity realization subject to \( L_2 \)-scaling constrains in closed form.

### REFERENCES


