Effects of Small Perturbation of the Difference of Phase Spectrums of Signals on Phase-only Correlation Functions

Jun ODAGIRI†, Shunsuke YAMAKI‡, Masahide ABE†, Masayuki KAWAMATA†
†Graduate School of Engineering
‡International Advanced Research and Education Organization
Tohoku University
Sendai, Japan
{odagiri, yamaki, masahide, kawamata}@mk.ecei.tohoku.ac.jp

Abstract—In this paper, we analyze properties of phase-only correlation functions. We analyze especially effects of small perturbation of the difference of phase spectrums of signals on phase-only correlation functions.

Keywords— phase-only signals, phase-only correlation functions

I. INTRODUCTION

A phase-only correlation function is useful for knowing a degree of similarity between two signals. Recently, it is used in many fields, such as image processing, pattern matching, etc. For example, it is used to estimate the frame displacement for old films [1] and applied to matching periodic DNA sequences [2].

The purpose of this paper is to show properties of a phase-only correlation function. Focusing on the difference of phase spectrums of signals, we analyze properties of phase-only correlation functions provided that:

• all of the difference of phase spectrums is zero except some difference $\alpha_p$ of phase spectrums
• the difference of phase spectrums is a small perturbation

II. PHASE-ONLY CORRELATION FUNCTIONS

A phase-only signal $\hat{x}(n)$ is the Inverse Discrete Fourier Transform of phase information of a signal $x(n)$ and is defined by

$$\hat{x}(n) = \frac{1}{N} \sum_{k=0}^{N-1} e^{j\theta_k} W_N^{-mk}, n = 0 \sim N - 1,$$

where $N$ is the size of a complex signal $x(n)$, $W_N$ is the twiddle factor $\exp\left(-j2\pi / N\right)$ and $\theta_k$ is a phase spectrum of a complex signal $x(n)$. Similarly, a phase-only signal $\hat{y}(n)$ with a phase spectrum $\phi_k$ is defined for a complex signal $y(n)$. Then, a phase-only correlation function between signals $x(n)$ and $y(n)$ is defined by

$$r_{xy}(m) = \sum_{n=0}^{N-1} \hat{x}(n)\hat{y}^*(n+m)_N$$

where $"\cdot"$ is a notation for the complex conjugate, $\alpha_k = \theta_k - \phi_k$ is the difference of phase spectrums of signals $x(n)$ and $y(n)$, and $(n + m)_N = n + m (mod N)$.

Consider the case that all of the difference of phase spectrums is zero, that is, $\alpha_k = 0$ for $k = 0 \sim N - 1$. The phase-only correlation function is given by

$$r_{xx}(m) = \frac{1}{N} \sum_{k=0}^{N-1} e^{j\alpha_k} W_N^{-mk} = \delta(m),$$

where $\delta(m)$ is the delta function. This equation is fundamental property.

III. THE ENERGY OF A PHASE-ONLY CORRELATION FUNCTION

According to Parseval’s theorem, the energy $E_{xy}$ of a phase-only correlation function is given by

$$E_{xy} = \sum_{m=0}^{N-1} r_{xy}(m) r_{xy}^*(m) = \frac{1}{N} \sum_{m=0}^{N-1} e^{j\alpha_k} (e^{j\alpha_k})^* = 1.$$

Let $r_{xy}(m) = a(m) + jb(m)$, where $a(m)$ and $b(m)$ are real numbers. Then, the range of the magnitude $|r_{xy}(m)|$ of the phase-only correlation and the range of $a(m)$ and $b(m)$ are given from (4) as follows:

$$|r_{xy}(m)| = \sqrt{a^2(m) + b^2(m)} \leq 1$$

$$-1 \leq a(m) \leq 1, -1 \leq b(m) \leq 1.$$

IV. THE EFFECT OF VARIATIONS OF THE DIFFERENCE OF PHASE SPECTRUMS ON A PHASE-ONLY CORRELATION FUNCTION

Consider an effect of the variation $\Delta\alpha_k$ of the difference of phase spectrums on a phase-only correlation function. The effect is given by the partial differential as follows:
\[
\frac{\partial \bar{r}(m)}{\partial \alpha_k} = \frac{j}{N} e^{j\alpha_k} W_{N}^{mk} \quad (7)
\]
\[
\left| \frac{\partial \bar{r}(m)}{\partial \alpha_k} \right| = \frac{1}{N}. \quad (8)
\]

The above equations show that the phase-only correlation function changes by \(\Delta \alpha_k/N\) with respect to the variation \(\Delta \alpha_k\) of the difference of phase spectrums. Furthermore, the magnitude of its variation does not depend on indices \(k\) and \(m\).

V. FOCUSING ON THE DIFFERENCE OF PHASE SPECTRUMS

In this section, we analyze properties of a phase-only correlation function, focusing on the difference of phase spectrums. When all of the difference of phase spectrums is zero, a phase-only correlation function is the delta function. This property is fundamental to know a degree of similarity of signals. However, all of the difference of phase spectrums is rarely zero. Therefore, we analyze variations of a phase-only correlation function when the difference of phase spectrums changes from zero.

A. all of the difference of phase spectrums is zero except some difference \(\alpha_p\) of phase spectrums

Suppose that all of the difference of phase spectrums is zero except some difference \(\alpha_p\) of phase spectrums, i.e., \(\alpha_k = 0\) except \(k \neq p\). Then, the phase-only correlation function is described as follows:

\[
r_{\bar{r}}(m) = \frac{1}{N} \sum_{k=0}^{N-1} e^{j\alpha_p} W_{N}^{-mk} = \delta(m) + \frac{1}{N} (e^{j\alpha_p} - 1) W_{N}^{-mp}. \quad (9)
\]

The above equation shows that the phase-only correlation function is the sum of the delta function (the first term of (9), called the main lobe) and the small noise (the second term of (9), called the side lobe).

In order to evaluate magnitudes of the phase-only correlation function, we consider the evaluation \(r_{\bar{r}}(m)r_{\bar{r}}(m)\) as follows:

\[
r_{\bar{r}}(m)r_{\bar{r}}(m) = \begin{cases} 
\frac{1}{N} - \frac{2}{N^2} (1 - \frac{1}{N}) (1 - \cos \alpha_p), & m = 0 \\
\frac{2}{N^2} (1 - \cos \alpha_p), & m = 1 \sim N - 1 
\end{cases} \quad (10)
\]

Equation (10) is the squared magnitude of the main lobe of the phase-only correlation function. Similarly, (11) is the squared magnitude of its side lobe. The main lobe and the side lobe depend only on some difference \(\alpha_p\) of phase spectrums but do not depend on an index \(p\). When \(\alpha_p = -\pi, \pi\), the magnitude of the main lobe is the minimum \(1 - 4(1 - 1/N)/N\), and the magnitude of the side lobe is the maximum \(4/N^2\). The energy of phase-only correlation functions is 1 from (4). Therefore, the main lobe is smaller as the side lobe is larger.

B. the difference of phase spectrums is a small perturbation

Let the difference of phase spectrums be \(\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_{N-1})\). Then, the phase-only correlation function (2) is denoted as follows:

\[
r_{\bar{r}}(m, \alpha) = \frac{1}{N} \sum_{k=0}^{N-1} e^{j\alpha_k} W_{N}^{-mk}. \quad (12)
\]

Let the perturbation of the difference of phase spectrums be \(\Delta \alpha = (\Delta \alpha_0, \Delta \alpha_1, \cdots, \Delta \alpha_{N-1})\). When the difference of phase spectrums changes from \(\alpha\) to \(\alpha + \Delta \alpha\), the Taylor series for the phase-only correlation function is denoted by

\[
r_{\bar{r}}(m, \alpha + \Delta \alpha) = r_{\bar{r}}(m, \alpha) + \sum_{p=0}^{N-1} \frac{\partial r_{\bar{r}}(m, \alpha)}{\partial \alpha_p} \Delta \alpha_p + \cdots. \quad (13)
\]

Then, the variation of the phase-only correlation function is defined as follows:

\[
\Delta r_{\bar{r}}(m, \alpha) = r_{\bar{r}}(m, \alpha + \Delta \alpha) - r_{\bar{r}}(m, \alpha) = \sum_{p=0}^{N-1} \frac{\partial r_{\bar{r}}(m, \alpha)}{\partial \alpha_p} \Delta \alpha_p + \cdots. \quad (14)
\]

If the perturbation \(\Delta \alpha\) of the difference of phase spectrums is small enough, higher order terms of (14) are very small. Therefore, the approximation of (14) is only the first order term:

\[
\Delta r_{\bar{r}}(m, \alpha) \approx \sum_{p=0}^{N-1} \frac{\partial r_{\bar{r}}(m, \alpha)}{\partial \alpha_p} \Delta \alpha_p. \quad (15)
\]

In order to evaluate the magnitude of the variation of a phase-only correlation function, we consider the evaluation \(\Delta r_{\bar{r}}(m, \alpha)\Delta r_{\bar{r}}(m, \alpha)\). The evaluation \(\Delta r_{\bar{r}}(m, \alpha)\Delta r_{\bar{r}}(m, \alpha)\) is described as follows:

\[
\Delta r_{\bar{r}}(m, \alpha)\Delta r_{\bar{r}}(m, \alpha) = \left(\sum_{p=0}^{N-1} \frac{\partial r_{\bar{r}}(m, \alpha)}{\partial \alpha_p} \Delta \alpha_p \right) \left(\sum_{p=0}^{N-1} \frac{\partial r_{\bar{r}}(m, \alpha)}{\partial \alpha_p} \Delta \alpha_p \right)^* \quad (16)
\]

Now, we use the inequality:

\[
(\xi_0 + \xi_1 + \cdots + \xi_{N-1})(\xi_0^* + \xi_1^* + \cdots + \xi_{N-1}^*) \leq N(\xi_0^* \xi_0^* + \xi_1^* \xi_1^* + \cdots + \xi_{N-1}^* \xi_{N-1}^*). \quad (17)
\]

The equality is attained if and only if \(\xi_0 = \xi_1 = \cdots = \xi_{N-1}\). According to (16) and (17), the evaluation is given by

\[
\Delta r_{\bar{r}}(m, \alpha)\Delta r_{\bar{r}}(m, \alpha) \leq \frac{1}{N} \sum_{p=0}^{N-1} \Delta \alpha_p^2. \quad (18)
\]

The equality (18) is attained if and only if \(\Delta \alpha_0 = \Delta \alpha_1 = \cdots = \Delta \alpha_{N-1}\). The inequality (18) shows that the evaluation \(\Delta r_{\bar{r}}(m, \alpha)\Delta r_{\bar{r}}(m, \alpha)\) is less than or equal to the mean squared value of the small perturbation \(\Delta \alpha_p\).

Figs. 1 and Figs. 2 show the squared magnitude of the variation of the phase-only correlation function. Small perturbation of the difference of phase spectrums is independent respectively and uniformly distributed. In Fig. 1, \(N = 4, m = 2\). In Fig. 2, \(N = 256, m = 128\). We can observe from Figs. 1 and Figs. 2 that the inequality (18) holds. However, in the case which the size \(N\) is large enough, the squared magnitude of the variation of the phase-only correlation function differs from the mean squared value of the small perturbation \(\Delta \alpha_p\).
VI. CONCLUSIONS

In this paper we analyze properties of a phase-only correlation function focusing on the difference of phase spectrums of signals. Our future subject is to find the better evaluation of a phase-only correlation function when the difference of phase spectrums is small perturbation.

REFERENCES


Figure 1. The squared magnitude of the variation of the phase-only correlation function ($N = 4$)

Figure 2. The squared magnitude of the variation of the phase-only correlation function ($N = 256$)