Closed Form Expressions of Balanced Realizations of Second-Order Analog Filters

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SUMMARY

This paper derives the balanced realizations of second-order analog filters directly from the transfer function. Second-order analog filters are categorized into the following three cases: complex conjugate poles, distinct real poles, and multiple real poles. For each case, simple formulas are derived for the synthesis of the balanced realizations of second-order analog filters. As a result, we obtain closed form expressions of the balanced realizations of second-order analog filters. Key words: state-space analog filters, balanced realizations, closed form expressions

1. Introduction

Balanced realizations are filter structures based on the state-space representation. It has been known that the balanced realizations have various ideal properties, which are useful in the field of signal processing \[1\]–\[6\]. They have already been shown to have minimum statistical coefficients sensitivity and to be free of limit cycles \[1\], and it can be adopted as an initial realization to minimize the $L_2$-sensitivity of state-space digital filters \[2\], \[3\]. Furthermore, model order reduction techniques based on balanced realizations have been investigated \[4\]–\[6\]. These techniques can be applied not only to digital filters, but also to analog filters. Therefore, balanced realizations have been widely used in many fields, such as digital signal processing, analog signal processing, and system control.

Closed form expressions of the balanced realizations, that is, synthesis of the balanced realization directly from a given transfer function, are quite useful since they greatly simplify the synthesis procedure of the balanced realizations. The authors previously proposed closed form expressions of second-order digital filters, which are often used as basic sections or sub-filters in implementation of high-order digital filters. For second-order digital filters with complex conjugate poles, Matsukawa and Kawamata derived a closed form expression of the balanced realization \[7\]. Furthermore, Yamaki, Abe, and Kawamata proposed closed form expressions of the balanced realizations of second-order digital filters with real poles \[8\].

However, for analog filters, there have been no attempts to give closed form expressions of the balanced realizations. It is strongly desired to derive similar algorithms for analog filters since the balanced realizations are quite important filter structures for analog filters as well as digital filters.

In this paper, we present closed form expressions of the balanced realizations of second-order analog filters. We show that closed form expressions of the balanced realizations for analog filters are quite similar to those of digital filters.

2. Preliminaries

2.1 Second-Order Analog Filters

Consider a stable second-order analog filter given by

$$H(s) = \frac{p_0 s^2 + p_1 s + p_2}{s^2 + q_1 s + q_2}.$$  

(1)

It is widely known that the analog filter (1) is stable if and only if $\text{Re}(\lambda_i) < 0$, where $\lambda_i (i = 1, 2)$ are poles of the filter (1). This condition is equivalent to the following conditions:

$$q_1 > 0 \text{ and } q_2 > 0.$$  

(2)

Figure 1 shows the stability region for second-order analog filters, which is quite similar than the stability triangle in the case of second-order digital filters \[8\]. For stable second-order analog filters (1), the types of the poles depend on the filter coefficients $q_1$ and $q_2$ as follows:

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig1.png}
\caption{Stability region of filter coefficients of second-order analog filters.}
\end{figure}
The balanced realization is the filter structure of which Gramians are equal and diagonal.

2.2 Balanced Realizations

Let $T$ be a nonsingular $2 \times 2$ real matrix. If a coordinate transformation defined by $\overline{x}(t) = T^{-1}x(t)$ is applied to a filter realization $(A, b, c, d)$, we obtain a new realization which has the following coefficient matrices

$$
\overline{A}, \overline{b}, \overline{c}, \overline{d} = (T^{-1}AT, T^{-1}b, cT, d).
$$

It should be noted that the transfer function $H(s)$ is invariant under the coordinate transformations since

$$
\overline{H}(s) = \overline{c}(sI - \overline{A})^{-1}\overline{b} + \overline{d} = eT(sI - T^{-1}AT)^{-1}T^{-1}b + d = eTT^{-1}(sI - A)^{-1}TT^{-1}b + d = e(sI - A)^{-1}b + d = H(s).
$$

On the other hand, the controllability and observability Gramians $(K_0, W_0)$ are transformed into $(\overline{K}_0, \overline{W}_0)$ under the coordinate transformation as follows:

$$
(\overline{K}_0, \overline{W}_0) = (T^{-1}K_0 T^{-T}, T^T W_0 T).
$$

Second order modes are invariant under the coordinate transformation since the eigenvalues of $\overline{K}_0 \overline{W}_0 = T^{-1}(K_0 W_0)T$ are equal to those of $K_b W_b$.

By the coordinate transformation, we can synthesize the balanced realization $(A_b, b_b, c_b, d_b)$, which is the filter structure whose controllability Gramian $K_b$ and observability Gramian $W_b$ are equal and diagonal as follows:

$$
K_b = W_b = \text{diag}(\theta_1, \theta_2).
$$

The balanced realization is used for the canonical form of the minimum statistical sensitivity realization [1].


In this section, we propose closed form expressions of the balanced realization of second-order analog filters. It is remarkable that closed form expressions of the balanced realizations for analog filters are quite similar to those of digital filters, although state equations and definitions of Gramians are different between digital filters and analog filters.
We define the scalar parameters $P$ as follows:

$$K_0^{(1)} = T_1^{-1} K_0^{(1)} T_1^T = \begin{bmatrix} P \cosh(2\alpha_1) - Q \sinh(2\alpha_1) + R & P \sinh(2\alpha_1) - Q \cosh(2\alpha_1) \\ P \sinh(2\alpha_1) - Q \cosh(2\alpha_1) & P \cosh(2\alpha_1) - Q \sinh(2\alpha_1) - R \end{bmatrix} \quad (22)$$

$$W_0^{(1)} = T_1^T W_0^{(1)} T_1 = \begin{bmatrix} P \cosh(2\alpha_1) - Q \sinh(2\alpha_1) - R & -P \sinh(2\alpha_1) + Q \cosh(2\alpha_1) \\ -P \sinh(2\alpha_1) + Q \cosh(2\alpha_1) & P \cosh(2\alpha_1) - Q \sinh(2\alpha_1) - R \end{bmatrix} \quad (23)$$

where $\kappa = \sqrt{\frac{P + Q}{P - Q}}$ and $\mu_1 = \frac{\sqrt{\kappa |\alpha| - \alpha_1}}{2}, \mu_2 = \frac{\sqrt{|\alpha| + \alpha_1}}{2\kappa} \text{sign}(\alpha_1)$.

### 3.1 Case 1: Two Poles are Complex Conjugate ($\alpha_1^2 - 4\alpha_2 < 0$)

We first consider a second-order analog filter $H_1(s)$ with coefficient matrices $(A_1^{(1)}, b_1^{(1)}, c_1^{(1)}, d_1^{(1)})$ whose poles are complex conjugates as follows:

$$H_1(s) = \frac{\alpha}{s - \lambda} + \frac{\alpha^*}{s - \lambda^*} + d$$

where $\alpha$ denotes complex conjugate, $\lambda = \lambda_1 + j\lambda_2$ is a complex pole, $\alpha = \alpha_1 + j\alpha_2$ is a complex scalar, and $d$ is a real scalar. We define the scalar parameters $P$, $Q$, and $R$ as follows:

$$P = -\frac{|\alpha|}{2\lambda}, \quad Q = \text{Im}\left(-\frac{\alpha}{2\lambda}\right), \quad R = \text{Re}\left(-\frac{\alpha}{2\lambda}\right).$$

We first determine the initial realization $(A_0^{(1)}, b_0^{(1)}, c_0^{(1)}, d_0^{(1)})$ of $H_1(s)$ as follows:

$$\begin{bmatrix} A_0^{(1)} \\ b_0^{(1)} \\ c_0^{(1)} \\ d_0^{(1)} \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_1 \\ -\lambda_1 & \lambda_1 \\ \sqrt{2\text{Re}(\alpha^*)} & -\sqrt{2\text{Im}(\alpha^*)} \\ \sqrt{2\text{Re}(\alpha^*)} & \sqrt{2\text{Im}(\alpha^*)} \end{bmatrix} \quad (18)$$

Substituting Eq. (18) into Eqs. (7) and (8), the controllability Gramian $K_0^{(1)}$ and the observability Gramian $W_0^{(1)}$ of the initial realization $(A_0^{(1)}, b_0^{(1)}, c_0^{(1)}, d_0^{(1)})$ are derived as

$$K_0^{(1)} = \begin{bmatrix} P + R & -Q \\ -Q & P - R \end{bmatrix} \quad (19)$$

$$W_0^{(1)} = \begin{bmatrix} P + R & Q \\ Q & P - R \end{bmatrix} \quad (20)$$

In order to diagonalize the Gramians $K_0^{(1)}$ and $W_0^{(1)}$, let $T_1$ be the coordinate transformation matrix described by

$$T_1 = \begin{bmatrix} \cosh(t_1) & -\sinh(t_1) \\ -\sinh(t_1) & \cosh(t_1) \end{bmatrix} \quad (21)$$

Under the coordinate transformation by $T_1$, the Gramians $K_0^{(1)}$ and $W_0^{(1)}$ are transformed into $K_0^{(1)}$ and $W_0^{(1)}$ such as Eqs. (22) and (23). We show that the Gramians $K_0^{(1)}$ and $W_0^{(1)}$ can be made equal and diagonal by specifying the parameter $t_1$ as

$$t_1 = \frac{1}{2} \tanh^{-1}\left(\frac{Q}{P}\right).$$

Substituting Eq. (24) into Eqs. (22) and (23) yields the controllability Gramian $K_0^{(1)}$ and the observability Gramian $W_0^{(1)}$ as follows:

$$K_0^{(1)} = W_0^{(1)} = \text{diag}(\theta_1^{(1)}, \theta_2^{(1)})$$

$$\theta_1^{(1)} = \sqrt{P^2 - Q^2} + R$$

$$\theta_2^{(1)} = \sqrt{P^2 - Q^2} - R.$$ 

Applying the coordinate transformation by $T_1$ with $t_1$ in Eq. (24) to $(A_0^{(1)}, b_0^{(1)}, c_0^{(1)}, d_0^{(1)})$, we derive the closed form expression of the balanced realization $(A_0^{(2)}, b_0^{(2)}, c_0^{(2)}, d_0^{(2)})$ as Eq. (28).

### 3.2 Case 2: Two Poles are Real and Distinct ($\alpha_1^2 - 4\alpha_2 > 0$)

We consider a second-order analog filter $H_2(s)$ with coefficient matrices $(A_2^{(2)}, b_2^{(2)}, c_2^{(2)}, d_2^{(2)})$ whose poles are real and distinct as follows:

$$H_2(s) = \frac{\alpha_1}{s - \lambda_1} + \frac{\alpha_2}{s - \lambda_2} + d$$

where $(\lambda_1, \lambda_2)$ are real poles, and $(\alpha_1, \alpha_2)$ are real scalars. We define the scalar parameters $P_1$, $P_2$, and $P_{12}$ as follows:

$$P_1 = -\frac{|\alpha_1|}{2\lambda_1}, \quad P_2 = -\frac{|\alpha_2|}{2\lambda_2}, \quad P_{12} = \frac{\sqrt{|\alpha_1\alpha_2|}}{\lambda_1 + \lambda_2}$$

It is obvious that $P_1 > 0$, $P_2 > 0$, and $P_{12} > 0$ since $\lambda_1$ and $\lambda_2$, the poles of $H_2(s)$, are negative for the stability. Without loss of generality, we assume $P_1 \geq P_2 > 0$. We first determine the initial realization $(A_0^{(2)}, b_0^{(2)}, c_0^{(2)}, d_0^{(2)})$ of $H_2(s)$ as follows:

$$\begin{bmatrix} A_0^{(2)} \\ b_0^{(2)} \\ c_0^{(2)} \\ d_0^{(2)} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \\ \sqrt{|\alpha_1|} \sigma_1 & \sqrt{|\alpha_2|} \sigma_2 \\ \sqrt{|\alpha_1|} \sigma_1 & \sqrt{|\alpha_2|} \sigma_2 \end{bmatrix}$$

(31)
Since we assume the argument of the function \( \tan^{-1} \) be the coordinate transformation matrix described by

\[
\sigma(\alpha) \text{ and } \sigma(\beta)
\]

show that the Gramians 

\[
K_{\alpha} = W_{\alpha}^{(2a)}
\]

Eq. (36) yields the controllability Gramian 

\[
K_{\alpha}(a)
\]

The case of 

\[
K_{\beta} = W_{\beta}^{(2a)}
\]

Applying the coordinate transformation by 

\[
T_{2a}
\]

with \( \phi \) in Eq. (37) to 

\[
(A_{\alpha}^{(2)}, b_{\alpha}^{(2)}, c_{\alpha}^{(2)}, d_{\alpha}^{(2)})
\]

we derive the closed form expression of the balanced realization 

\[
(A_{b}^{(2)}, b_{b}^{(2)}, c_{b}^{(2)}, d_{b}^{(2)})
\]

as Eq. (41) where \( \sigma = \sigma_{1} = \sigma_{2} \).

(b) The case of \( \sigma_{1} = -\sigma_{2} \)

In this case, the controllability and observability Gramians are given by

\[
K_{\beta}^{(2b)} = W_{\beta}^{(2b)} = \text{diag}(\theta_{1}^{(2b)}, \theta_{2}^{(2b)})
\]

\[
\theta_{1}^{(2b)} = \frac{1}{2}(P_{1} + P_{2}) + \frac{1}{2}\sqrt{(P_{1} - P_{2})^2 + 4P_{12}^2}
\]

\[
\theta_{2}^{(2b)} = \frac{1}{2}(P_{1} + P_{2}) - \frac{1}{2}\sqrt{(P_{1} - P_{2})^2 + 4P_{12}^2}
\]

In order to diagonalize the Gramians 

\[
K_{\alpha}^{(2b)} = W_{\alpha}^{(2b)}
\]

be the coordinate transformation matrix described by

\[
T_{2b} = \begin{bmatrix}
\cosh(t_2) & \sinh(t_2) \\
\sinh(t_2) & \cosh(t_2)
\end{bmatrix}
\]

Under the coordinate transformation by the orthogonal matrix 

\[
T_{2b}
\]

the Gramians 

\[
K_{\alpha}^{(2b)} = W_{\alpha}^{(2b)}
\]

are transformed into 

\[
\begin{bmatrix}
P_{1} & P_{12} \\
P_{12} & P_{2}
\end{bmatrix}
\]

In order to diagonalize the Gramians 

\[
K_{\alpha}^{(2b)} = W_{\alpha}^{(2b)}
\]

we show that the Gramians 

\[
K_{\alpha}^{(2b)} = W_{\alpha}^{(2b)}
\]

can be made equal and diagonal by specifying the parameter \( \phi \) as

\[
\phi = \frac{1}{2}\tan^{-1}\left(\frac{2P_{12}}{P_{1} - P_{2}}\right)
\]

(37)

Since we assume \( P_{1} \geq P_{2} \) and \( P_{12} > 0 \), the range of the argument of the function \( \tan^{-1} \) is \((0, \infty)\), which yields the range of \( \phi \) given in Eq. (37). Substituting Eq. (37) into Eq. (36) yields the controllability Gramian 

\[
K_{\beta}^{(2b)} = W_{\beta}^{(2b)} = \text{diag}(\theta_{1}^{(2b)}, \theta_{2}^{(2b)})
\]

\[
\theta_{1}^{(2b)} = \frac{1}{2}\sqrt{(P_{1} + P_{2})^2 - 4P_{12}^2} + \frac{1}{2}(P_{1} - P_{2})
\]

\[
\theta_{2}^{(2b)} = \frac{1}{2}\sqrt{(P_{1} + P_{2})^2 - 4P_{12}^2} - \frac{1}{2}(P_{1} - P_{2})
\]
Applying the coordinate transformation by $T_{2b}$ with $t_2$ in Eq. (47) to $(A_0^{(2b)}, b_0^{(2b)}, c_0^{(2b)}, d_0^{(2b)})$, we derive the closed form expression of balanced realization $(A_0^{(2b)}, b_0^{(2b)}, c_0^{(2b)}, d_0^{(2b)})$ as Eq. (51).

### 3.3 Case 3: Two Poles Are Real and Multiple ($q_0^2 - 4q_2 = 0$)

We consider a second-order analog filter $H_3(s)$ with coefficient matrices $(A_0^{(3)}, b_0^{(3)}, c_0^{(3)}, d_0^{(3)})$ whose poles are real and multiple as follows:

$$H_3(s) = \frac{\beta_1}{s - \lambda_0} + \frac{\beta_2}{(s - \lambda_0)^2} + d$$

where $\lambda_0$ is a real double pole, and $(\beta_1, \beta_2)$ are real scalars. We define the scalar parameters $Q_1$, $Q_2$, and $Q_{12}$ as follows:

$$Q_1 = -\frac{\beta_2}{4|\beta_2|} + \frac{\sigma |\beta_2|}{2|\beta_2|}$$

$$Q_2 = -\frac{|\beta_2|}{2\lambda_0}$$

$$Q_{12} = -\frac{\beta_1}{2\lambda_0} + \frac{|\beta_2|}{\lambda_0}$$

where $\sigma = \text{sign}(\beta_2)$. We first determine the initial realization $(A_0^{(3)}, b_0^{(3)}, c_0^{(3)}, d_0^{(3)})$ of $H_3(s)$ as follows:

$$\begin{bmatrix} A_0^{(3)} & b_0^{(3)} \\ c_0^{(3)} & d_0^{(3)} \end{bmatrix} \begin{bmatrix} A_0^{(3)} & b_0^{(3)} \\ c_0^{(3)} & d_0^{(3)} \end{bmatrix}^{-1} \begin{bmatrix} T_{2b}^{-1} & 0 \\ 0 & T_{2b}^{-1} \end{bmatrix} \begin{bmatrix} T_{2b}^{-1} & 0 \\ 0 & T_{2b}^{-1} \end{bmatrix}$$

Substituting Eq. (56) into Eqs. (7) and (8), the controllability Gramian $K_0^{(3)}$ and the observability Gramian $W_0^{(3)}$ of the initial realization $(A_0^{(3)}, b_0^{(3)}, c_0^{(3)}, d_0^{(3)})$ are derived as

$$K_0^{(3)} = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12} & Q_2 \end{bmatrix}, \quad W_0^{(3)} = \begin{bmatrix} Q_3 & Q_{12} \\ Q_{12} & Q_1 \end{bmatrix}.$$
\[
T_3 = \frac{1}{\sqrt{2}} \begin{bmatrix}
 t_3 & -t_3 \\
 t_3^{-1} & t_3^{-1}
\end{bmatrix}.
\] (58)

Under the coordinate transformation by \( T_3 \), the Gramians \( K_b^{(3)} \) and \( W_b^{(3)} \) are transformed into \( \overline{K}_b^{(3)} \) and \( \overline{W}_b^{(3)} \) such as Eqs. (59) and (60). We show that the Gramians \( \overline{K}_b^{(3)} \) and \( \overline{W}_b^{(3)} \) can be made equal and diagonal by specifying the parameter \( t_3 \) as

\[
t_3 = \left( \frac{Q_1}{Q_2} \right)^{1/2}. \] (61)

Substituting Eq. (61) into Eqs. (59) and (60) yields the controllability Gramian \( K_b^{(3)} \) and the observability Gramian \( W_b^{(3)} \) as follows:

\[
K_b^{(3)} = W_b^{(3)} = \text{diag}(\theta_1^{(3)}, \theta_2^{(3)}) \] (62)

\[
\theta_1^{(3)} = \sqrt{Q_1Q_2 + Q_{12}} \] (63)

\[
\theta_2^{(3)} = \sqrt{Q_1Q_2 - Q_{12}}. \] (64)

Applying the coordinate transformation by \( T_3 \) with \( t_3 \) in Eq. (61) to \( (A_3^{(3)}, b_3^{(3)}, c_0^{(3)}, d_0^{(3)}) \), we derive the closed form expression of the balanced realization \( (A_b^{(3)}, b_b^{(3)}, c_b^{(3)}, d_b^{(3)}) \) as Eq. (65).

4. Conclusions

This paper has proposed closed form expressions of balanced realizations of second-order analog filters. We have derived closed form expressions of the balanced realizations of three types of second-order analog filters; filters of complex conjugate poles, real and distinct poles, and real and multiple poles. Our formulation enables us to compute the coefficient matrices of the balanced realizations directly from the transfer function. It must be noted that the closed form expressions of the balanced realization of second-order analog filters are quite similar to those of second-order digital filters.

References


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