Summary

This letter proposes closed form solutions to the \( L_2 \)-sensitivity minimization of second-order state-space digital filters with real poles. We consider two cases of second-order digital filters: distinct real poles and multiple real poles. In case of second-order digital filters, we can express the \( L_2 \)-sensitivity of second-order digital filters by a simple linear combination of exponential functions and formulate the \( L_2 \)-sensitivity minimization problem by a simple polynomial equation. As a result, the minimum \( L_2 \)-sensitivity realizations can be synthesized by only solving a fourth-degree polynomial equation, which can be analytically solved.

Key words: state-space digital filters, \( L_2 \)-sensitivity, minimization, closed form solutions

1. Introduction

\( L_2 \)-sensitivity is one of the evaluation functions which evaluate the coefficient quantization effects of state-space digital filters \([1]–[3]\). The \( L_2 \)-sensitivity minimization is quite beneficial technique for the synthesis of high-accuracy digital filter structures, which achieve quite low-coefficient quantization error.

To the \( L_2 \)-sensitivity minimization problem, Yan et al. \([1]\) and Hinamoto et al. \([2]\) proposed solutions using iterative calculations. Both of the solutions in \([1]\) and \([2]\) try to solve nonlinear equations by successive approximation. Our group previously derived a closed form solution to the \( L_2 \)-sensitivity minimization problem of second-order digital filters \([3]\). Second-order digital filters play important role in implementation of higher-order digital filters as basic sections or sub-filters \([4]–[6]\). Thus, second-order digital filters are useful and important realizations in considering the \( L_2 \)-sensitivity minimization. However, the closed form solution we proposed in \([3]\) is not applicable to second-order digital filters with real poles while it is applicable to second-order digital filters with complex conjugate poles.

This letter is an extension of the method we proposed in \([3]\). We present closed form solutions to the \( L_2 \)-sensitivity minimization problem for second-order digital filters with real poles. Actually, second-order digital filters with real poles cover a large region in stability triangle, as shown in Fig. 1. Therefore, it is also necessary to derive the closed form solutions to the \( L_2 \)-sensitivity minimization for second-order digital filters with real poles. As a result of this letter, we will be able to cover all types of second-order digital filters for the synthesis of the minimum \( L_2 \)-sensitivity realizations.

2. Preliminaries

2.1 Second-Order Digital Filters

Consider a stable second-order IIR digital filter given by

\[
H(z) = \frac{p_0 + p_1 z^{-1} + p_2 z^{-2}}{1 + q_1 z^{-1} + q_2 z^{-2}}. \tag{1}
\]

It is well known that the second-order digital filter \( H(z) \) is stable, if and only if \( q_1 \) and \( q_2 \) stay within the stability triangle described by

\[
|q_1| < 1, \ |q_1| < 1 + q_2. \tag{2}
\]

Figure 1 shows the stability triangle. For stable second-order digital filters given by (1), the locations of the poles depend on the filter coefficients \( q_1 \) and \( q_2 \) as follows:

- Case 1: Poles are complex conjugate if \( q_1^2 - 4q_2 < 0 \).
- Case 2: Poles are real and distinct if \( q_1^2 - 4q_2 > 0 \). \tag{3}
- Case 3: Poles are real and multiple if \( q_1^2 - 4q_2 = 0 \).

We synthesize the minimum \( L_2 \)-sensitivity realizations by the state-space approach. The second-order digital filter

\[
\text{Fig. 1 Stability triangle of second-order digital filters.}
\]
(1) can be described by the following state-space representation:
\[
\begin{align*}
x(n + 1) &= Ax(n) + bu(n) \\
y(n) &= cx(n) + du(n)
\end{align*}
\]
(4) (5)
where \(x(n) = [x_1(n)x_2(n)]^T\) is a state-vector, \(u(n) \in \mathbb{R}\) is a scalar input, \(y(n) \in \mathbb{R}\) is a scalar output, and \(A \in \mathbb{R}^{2 \times 2}\), \(b \in \mathbb{R}^{2 \times 1}\), \(c \in \mathbb{R}^{1 \times 2}\), \(d \in \mathbb{R}\) are real constant matrices called coefficient matrices. The transfer function \(H(z)\) is described in terms of the coefficient matrices \((A, b, c, d)\) as \(H(z) = c(zI - A)^{-1}b + d\).

2.2 \(L_2\)-Sensitivity

The \(L_2\)-sensitivity is one of the measurements which evaluate coefficient quantization effects of digital filters. The \(L_2\)-sensitivity of the filter \(H(z)\) with respect to the realization \((A, b, c, d)\) is defined by using the general controllability Gramian \(K_i\) and the general observability Gramian \(W_i\) such as [2]
\[
S(A, b, c) = \left\| \frac{\partial H(z)}{\partial A} \right\|_2^2 + \left\| \frac{\partial H(z)}{\partial b} \right\|_2^2 + \left\| \frac{\partial H(z)}{\partial c} \right\|_2^2
\]
(6)
\[
= \text{tr}(W_i)\text{tr}(K_i) + \text{tr}(W_0) + \text{tr}(K_0)
\]
\[
+ 2 \sum_{i=1}^{\infty} \text{tr}(W_i)\text{tr}(K_i).
\]

The general Gramians \(K_i\) and \(W_i\) are defined as solutions to the Lyapunov equations expressed as
\[
K_i = AK_iA^T + \frac{1}{2} \left( A^Tbb^T + bb^T(A^T)^i \right)
\]
(7)
\[
W_i = A^TW_iA + \frac{1}{2} \left( c^TcA^i + (A^T)^ic^Tc \right)
\]
(8)
for \(i = 0, 1, 2, \ldots\), respectively. By simple mathematical manipulation, we derive the novel expressions of general Gramians from Eqs. (7) and (8) as follows:
\[
K_i = \frac{1}{2} \left( A^iK_0 + K_0(A^T)^i \right)
\]
(9)
\[
W_i = \frac{1}{2} \left( W_0A^i + (A^T)^iW_0 \right).
\]
(10)
The controllability Gramian \(K_0\) and the observability Gramian \(W_0\) are obtained by letting \(i = 0\) in Eqs. (7) and (8) as follows:
\[
K_0 = AK_0A^T + bb^T
\]
(11)
\[
W_0 = A^TW_0A + c^Tc.
\]
(12)

2.3 Coordinate Transformation

Let \(T\) be a nonsingular \(2 \times 2\) real matrix. If a coordinate transformation defined by \(\tilde{x}(n) = T^{-1}x(n)\) is applied to a filter realization \((A, b, c, d)\), we obtain a new realization which has the following coefficient matrices
\[
(\tilde{A}, \tilde{b}, \tilde{c}, \tilde{d}) = (T^{-1}AT, T^{-1}b, cT, d)
\]
(13)
and the following general Gramians
\[
(\tilde{K}_i, \tilde{W}_i) = (T^{-1}K_iT^{-T}, T^{-1}W_iT)
\]
(14)
respectively. The \(L_2\)-sensitivity of the transformed filter \((T^{-1}AT, T^{-1}b, cT, d)\) can be expressed in terms of the infinite summation of general Gramians as
\[
S(P) = \text{tr}(W_0P)\text{tr}(K_0P^{-1}) + \text{tr}(W_0P) + \text{tr}(K_0P^{-1})
\]
\[
+ 2 \sum_{i=1}^{\infty} \text{tr}(W_iP)\text{tr}(K_iP^{-1})
\]
(15)
where \(P\) is a positive definite symmetric matrix defined by \(P = TT^T\) [2].

2.4 \(L_2\)-Sensitivity Minimization Problem

We call the positive definite symmetric matrix which gives the global minimum of \(S(P)\) the optimal positive definite symmetric matrix \(P_{\text{opt}}\). The problem we consider here is to derive the optimal positive definite symmetric matrix \(P_{\text{opt}}\). Consequently, the \(L_2\)-sensitivity minimization problem is formulated as follows: For an initial digital filter \((A, b, c, d)\) with a given transfer function \(H(z)\), minimize the \(L_2\)-sensitivity \(S(P)\) with respect to \(P\), where \(P\) is an arbitrary positive definite symmetric matrix.

References [1] reports that \(L_2\)-sensitivity \(S(P)\) has the unique global minimum, which is achieved by \(P_{\text{opt}}\) satisfying
\[
\frac{\partial S(P)}{\partial P}\bigg|_{P=P_{\text{opt}}} = 0.
\]
(16)
Several approaches to solve the differential Eq. (16) are proposed [1],[2]. The algorithm proposed by Yan et al. [1] is a successive approximation using Riccati difference equation. The algorithm proposed by Hinamoto et al. [2] is a successive approximation without step size parameter. Both of these algorithms require the iterative calculations to derive the minimum \(L_2\)-sensitivity realization. On the other hand, our proposed method derives the minimum \(L_2\)-sensitivity realization in closed form without iterative calculations.

3. Balanced Realizations of Second-Order Digital Filters

This section introduces the balanced realization of second-order digital filters. It is expedient to adopt a balanced realization as the initial realization in order to synthesize the minimum \(L_2\)-sensitivity realization. We exploit the symmetric properties of the balanced realization in order to simplify the \(L_2\)-sensitivity formulation and minimization.

The balanced realization \((A_b, b_b, c_b, d_b)\) is the filter realization of which controllability and observability Gramians are diagonal and equal as follows:
\[
K_0 = W_0 = \Theta, \quad \Theta = \text{diag}(\theta_1, \theta_2)
\]
(17)
where the parameters $\theta_i (i = 1, 2)$ are the second-order modes of the filter $H(z)$. Substituting Eq. (17) into Eqs. (9) and (10), we can express the general Gramians of the balanced realization $(A_b, b_b, c_b, d_b)$ as follows:

$$K_i = \frac{1}{2} \left( A_b^i \Theta + \Theta (A_b^T)^i \right)$$ (18)

$$W_i = \frac{1}{2} \left( \Theta A_b^i + (A_b^T)^i \Theta \right).$$ (19)

The $L_2$-sensitivity $S(P)$ of a filter $(T^{-1} A_b T, T^{-1} b_b, c_b T, d_b)$ is expressed by substituting Eqs. (17), (18), and (19) into Eq. (15) as follows:

$$S(P) = \text{tr}(\Theta P \text{tr}(\Theta P^{-1}) + \text{tr}(\Theta P) + \text{tr}(\Theta P^{-1})$$

$$+ 2 \sum_{i=1}^{\infty} \text{tr}(\Theta A_b^i P) \text{tr}(\Theta (A_b^T)^i P^{-1}).$$ (20)

In order to derive closed form expression of the $L_2$-sensitivity $S(P)$, it is necessary to give closed form expressions of $A_b$, $\Theta$, and $P$, and substitute them into Eq. (20). The following subsections reviews the closed form expressions of the balanced realizations of second-order digital filters, which give the closed form expressions of $A_b$ and $\Theta$.

### 3.1 Case 1: Complex Conjugate Poles [6]

For second-order digital filters with complex conjugate poles, our group previously derived a closed form expression of the balanced realization. Second-order digital filters whose poles are complex conjugate are defined as follows:

$$H(z) = \frac{\alpha}{z - \alpha^*} + \frac{\alpha^*}{z - \alpha} + d$$ (21)

where $\alpha = \lambda_i + j\lambda_i$ is a complex pole, $\alpha = \alpha_i + j\alpha_i$ is a complex number and $d$ is a real number. We define real parameters $P$, $Q$, and $R$ as follows:

$$P = \frac{|\alpha|}{1 - |\alpha|^2}, \quad R + jQ = \frac{\alpha}{1 - \lambda^2}$$ (22)

which can be calculated directly from the transfer function $H(z)$. The closed form expression of the coefficient matrix $A_b$ is given by:

$$A_b = \begin{bmatrix} \lambda_i - \frac{\kappa - \kappa^{-1}}{2} \lambda_i & \frac{\kappa + \kappa^{-1}}{2} \lambda_i \\ -\frac{\kappa + \kappa^{-1}}{2} \lambda_i & \lambda_i + \frac{\kappa - \kappa^{-1}}{2} \lambda_i \end{bmatrix}$$ (23)

where

$$\kappa = \sqrt{\frac{P + Q}{P - Q}}.$$ (24)

Using the parameters $P$, $Q$, and $R$, the controllability Gramian $K_0$ and the observability Gramian $W_0$ of the balanced realization can be expressed as follows:

$$K_0 = W_0 = \Theta$$ (25)

$$\Theta = \text{diag}(\theta_1, \theta_2)$$

$$= \text{diag}(\sqrt{P^2 - Q^2} + R, \sqrt{Q^2 - P^2} - R).$$ (26)

### 3.2 Case 2: Real and Distinct Poles [7]

We consider second-order digital filters whose poles are real and distinct as follows:

$$H(z) = \frac{\alpha_1}{z - \lambda_1} + \frac{\alpha_2}{z - \lambda_2} + d \quad (\lambda_1, \lambda_2)$$ (27)

where $(\lambda_1, \lambda_2)$ are real poles, $(\alpha_1, \alpha_2)$ are real scalars, and $d$ is a real scalar. We define the scalar parameters $P_1, P_2$, and $P_{12}$ as follows:

$$P_1 = \frac{|\alpha_1|}{1 - \lambda_1^2}, \quad P_2 = \frac{|\alpha_2|}{1 - \lambda_2^2}, \quad P_{12} = \frac{\sqrt{|\alpha_1| |\alpha_2|}}{1 - \lambda_1 \lambda_2}$$ (28)

which can be calculated directly from the transfer function $H(z)$. It is obvious that $P_1 > 0$, $P_2 > 0$, and $P_{12} > 0$. Without loss of generality, we assume $P_1 \geq P_2 > 0$.

The coefficient matrices of the balanced realization depend on the signs $\alpha_1 = \text{sign}(\alpha_1)$ and $\alpha_2 = \text{sign}(\alpha_2)$. We consider two cases of the signs: (a) $\alpha_1 = \alpha_2$ and (b) $\alpha_1 \neq \alpha_2$.

#### Case 2(a): $\alpha_1 = \alpha_2$

The closed form expression of the coefficient matrix $A_b$ is given by Eq. (29). The controllability Gramian $K_0$ and the observability Gramian $W_0$ of the balanced realization are expressed as follows:

$$K_0 = W_0 = \text{diag}(\theta_1, \theta_2)$$ (30)

$$\theta_1 = \frac{1}{2}(P_1 + P_2) + \frac{1}{2} \sqrt{(P_1 - P_2)^2 + 4P_{12}^2}$$ (31)

$$\theta_2 = \frac{1}{2}(P_1 + P_2) - \frac{1}{2} \sqrt{(P_1 - P_2)^2 + 4P_{12}^2}.$$ (32)

#### Case 2(b): $\alpha_1 \neq \alpha_2$

The closed form expression of the coefficient matrix $A_b$ is given by Eq. (33). The controllability Gramian $K_0$ and the observability Gramian $W_0$ of the balanced realization are
expressed as follows:

\[ K_0 = W_0 = \text{diag}(\theta_1, \theta_2) \]  
\[ \theta_1 = \frac{1}{2} \sqrt{(P_1 + P_2)^2 - 4P^2_{12}} + \frac{1}{2}(P_1 - P_2) \]  
\[ \theta_2 = \frac{1}{2} \sqrt{(P_1 + P_2)^2 - 4P^2_{12}} - \frac{1}{2}(P_1 - P_2). \]  

3.3 Case 3: Real and Multiple Poles [7]

We consider second-order digital filters whose poles are real and multiple as follows:

\[ H_3(z) = \frac{\beta_1}{z - \lambda_0} + \frac{\beta_2}{(z - \lambda_0)^2} + d \]  

where \( \lambda_0 \) is a real double pole, \((\beta_1, \beta_2)\) are real scalars, and \( d \) is a real scalar. We define the scalar parameters \( Q_1, Q_2, \) and \( Q_{12}, \) which can be calculated directly from the transfer function \( H_3(z), \) as follows:

\[ Q_1 = \frac{\beta^2_1}{4|\beta_2|} + \frac{1}{(1 - \lambda_0^2)} + \frac{\alpha |\beta_1| \lambda_0}{(1 - \lambda_0^2)^2} \frac{1 + \lambda^2_0}{(1 - \lambda_0^2)^2} \]  
\[ Q_2 = \frac{|\beta_2|}{1 - \lambda_0^2} \]  
\[ Q_{12} = \frac{1}{2} \left( \frac{\sigma \beta_1}{1 - \lambda_0^2} + \frac{1 + \lambda^2_0}{(1 - \lambda_0^2)^2} \right) \]  

where \( \sigma = \text{sign}(\beta_2). \) The closed form expression of the coefficient matrix \( A_b \) is given by

\[ A_b = \begin{bmatrix} \lambda_0 + \frac{1}{2} r^{-2} & \frac{1}{2} r^{-2} \\ -\frac{1}{2} r^{-2} & \lambda_0 - \frac{1}{2} r^{-2} \end{bmatrix}, \quad r = \left( \frac{Q_1}{Q_2} \right)^{\frac{1}{2}}. \]  

The controllability Gramian \( K_0 \) and the observability Gramian \( W_0 \) of the balanced realization are expressed as follows:

\[ K_0 = W_0 = \text{diag}(\theta_1, \theta_2) \]  
\[ \theta_1 = \sqrt{Q_1/Q_2} + Q_{12} \]  
\[ \theta_2 = \sqrt{Q_1/Q_2} - Q_{12}. \]  

4. Closed Form Solutions to the \( L_2 \)-Sensitivity Minimization

In this section, we newly propose closed form solutions to the \( L_2 \)-sensitivity minimization problem.

4.1 Closed Form Expressions of Matrix \( P \)

In the previous chapter, the closed form expressions of \( A_b \) and \( \Theta \) are given. In order to derive the closed form expression of the \( L_2 \)-sensitivity \( S(P) \) in Eq. (20), it is necessary to derive the closed form expression of the positive definite symmetric matrix \( P. \)

The closed form expressions of positive definite symmetric matrix \( P \) are determined by symmetric properties of coefficient matrices of the balanced realizations. The symmetric properties of synthesized balanced realizations \((A_b, b_0, c_0, d_b)\) are described by the sign matrix \( \Sigma \) as follows:

\[ A^T_b = \Sigma A_b \Sigma, \quad c^T_b = \Sigma b_b. \]  

For second-order digital filters we have discussed in this section, we clarify the form of the sign matrix \( \Sigma \) as follows:

\[ \text{Case 1} : H_1(z) = \frac{\alpha}{z - \lambda} + \frac{\alpha^*}{z - \lambda^*} + d \]  
\[ \Sigma = \text{diag}(1, -1). \]  
\[ \text{Case 2} : H_2(z) = \frac{\alpha_1}{z - \lambda_1} + \frac{\alpha_2}{z - \lambda_2} + d \]  
\[ (a) \text{ If } \sigma_1 = \sigma_2, \quad \Sigma = \pm I. \]  
\[ (b) \text{ If } \sigma_1 \neq \sigma_2, \quad \Sigma = \text{diag}(\sigma_1, \sigma_2) = \pm \text{diag}(1, -1). \]  
\[ \text{Case 3} : H_3(z) = \frac{\alpha_1}{z - \lambda_0} + \frac{\alpha_2}{(z - \lambda_0)^2} + d \]  
\[ \Sigma = \text{diag}(\sigma, -\sigma) = \pm \text{diag}(1, -1). \]

The authors previously proved in Ref. [3] the positive definite symmetric matrix \( P \) which minimizes the \( L_2 \)-sensitivity \( S(P) \) in (20) is given as follows:

\[ P = \begin{bmatrix} \cosh(p) & \sinh(p) \\ \sinh(p) & \cosh(p) \end{bmatrix}, \quad \text{if } \Sigma = \pm \text{diag}(1, -1). \]  
\[ P = I, \quad \text{if } \Sigma = \pm \text{diag}(1, 1). \]

Therefore, Eq. (46) reveals that the closed form expressions of the positive definite symmetric matrices \( P \) are categorized as follows:

\[ P = \begin{bmatrix} \cosh(p) & \sinh(p) \\ \sinh(p) & \cosh(p) \end{bmatrix} \text{ for Case 1, 2(b), and 3.} \]  
\[ P = I \text{ for Case 2(a).} \]
4.2 Closed Form Expressions of $L_2$-Sensitivity $S(P)$

We newly derive closed form expression of the $L_2$-sensitivity $S(P)$. For Case 1, we have already derived closed form expression of the $L_2$-sensitivity in Ref. [3]. For Case 2(a), the minimum $L_2$-sensitivity realization is equal to the balanced realization since $P = I$. Therefore, it is enough to derive the closed form expression of the $L_2$-sensitivity $S(P)$ for Case 2(b) and Case 3.

The closed form expression of the positive definite matrix $P$ in Eq. (48) enables us to convert $S(P)$ in Eq. (20) into a function of the single scalar variable $S(p)$. Substituting the closed form expressions of $A_b$, $\Theta$, and $P$ into Eq. (20) gives the closed form expression of the $L_2$-sensitivity $S(P)$ as

$$ S(P) = S(p) = \sum_{n=-2}^{2} s_n e^{\alpha n} $$

(49)

where $s_n$'s($n = -2, -1, 0, 1, 2$) are given by Eqs. (50)–(54) for Case 2(b) and Eqs. (55)–(59) for Case 3, respectively. It is remarkable that Eq. (49) is a simple linear combination of exponential functions which does not contain infinite summation. These coefficients $s_n$'s are easily computed directly from the transfer function $H(z)$ in Eq. (27) for Case 2(b) and Eq. (37) for Case 3, respectively. We thus realize the closed form expression of the $L_2$-sensitivity of second-order digital filters.

4.3 Synthesis of Minimum $L_2$-Sensitivity Realizations of Second-Order Digital Filters

The scalar parameter $p$ which minimizes $S(p)$ can be derived by solving the following equation with respect to $p$:

$$ \frac{\partial S(p)}{\partial p} = \sum_{n=-2}^{2} n s_n e^{\alpha n} = 0. $$

(60)

Letting $\beta = e^p$ gives

$$ \sum_{n=-2}^{2} n s_n \beta^p = 0. $$

(61)

The above equation is a fourth-degree polynomial equation, which can be analytically solved, with respect to $\beta$ of constant coefficients. Equation (61) has four solutions, from which the positive real solution $\beta_{\text{opt}} = e^{\alpha_1}$ is adopted to derive the optimal positive definite symmetric matrix $P_{\text{opt}}$ as

$$ P_{\text{opt}} = \begin{bmatrix} \cosh(\beta_{\text{opt}}) & \sinh(\beta_{\text{opt}}) \\ \sinh(\beta_{\text{opt}}) & \cosh(\beta_{\text{opt}}) \end{bmatrix} $$

(62)

The diagonalization of the optimal positive definite symmetric matrix $P_{\text{opt}} = T_{\text{opt}} P_{\text{opt}} T_{\text{opt}}^T$ is given by
Once the optimal positive definite symmetric matrix $P_{\text{opt}}$ is derived, the optimal coordinate transformation matrix $T_{\text{opt}}$ is calculated as

$$T_{\text{opt}} = P_{\text{opt}}^{1/2}U$$

where $U$ is an arbitrary orthogonal matrix. We have proved in Ref.[8] that specifying the orthogonal matrix by $U = R^T$ gives the minimum $L_2$-sensitivity without limit cycles. Therefore, we let $U = R^T$ in Eq.(64), and determine the optimal coordinate transformation matrix $T_{\text{opt}}$ as follows:

$$T_{\text{opt}} = P_{\text{opt}}^{1/2}R^T = R^TB^{1/2}$$

where $R = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

Applying the coordinate transformation by $T_{\text{opt}}$ to the balanced realization $(A_b, b_b, c_b, d_b)$ yields the minimum $L_2$-sensitivity realization $(A_{\text{opt}}, b_{\text{opt}}, c_{\text{opt}}, d_{\text{opt}})$ as follows:

$$\begin{bmatrix} A_{\text{opt}} \\ b_{\text{opt}} \\ c_{\text{opt}} \\ d_{\text{opt}} \end{bmatrix} = \begin{bmatrix} T_{\text{opt}}^{-1}A_bT_{\text{opt}} & T_{\text{opt}}^{-1}b_b \\ c_{\text{opt}} & d_{\text{opt}} \end{bmatrix}.$$  

We can give the closed form expression of the minimum $L_2$-sensitivity realization $(A_{\text{opt}}, b_{\text{opt}}, c_{\text{opt}}, d_{\text{opt}})$ as Eq.(67) for Case 2(b) and Eq.(68) for Case 3.

5. Conclusion

This letter has proposed minimization of $L_2$-sensitivity in case of second-order digital filters with real poles. The minimum $L_2$-sensitivity realization can be derived by only solving a fourth-degree polynomial equation of constant-coefficients as well as the case of second-order digital filters with complex conjugate poles. As a result of this letter, we can cover all types of second-order digital filters for the synthesis of the minimum $L_2$-sensitivity realizations.

References


