EFFECTS OF STOCHASTIC PHASE-SPECTRUM DIFFERENCES ON PHASE-ONLY CORRELATION FUNCTIONS

— PART II: STATISTICALLY PROPORTIONAL PHASE SPECTRUM DIFFERENCES TO FREQUENCY INDICES —

Rihito Ito, Shunsuke Yamaki, Masahide Abe, Masayuki Kawamata

Department of Electronic Engineering, Graduate School of Engineering, Tohoku University, Sendai, Japan

rihito@mk.ecei.tohoku.ac.jp

Abstract: This paper discusses properties of the phase-only correlation (POC) functions for stochastic phase spectrum differences. Assuming the phase spectrum differences between two signals to be random variables, we derive general expressions of the expectation and variance of the POC functions and show that the expectation of the peak of the POC functions decreases and the variances of the POC functions increase as the phase spectrum differences increase. This property mathematically guarantees validity of the POC functions. We will next discuss validity of the POC functions for evaluating similarity between two signals.

Keywords: Phase-only correlation; Similarity measure; Stochastic phase spectrum difference

1 Introduction

The phase-only correlation (POC) functions have been widely known as an effective method for evaluating similarity between two signals. They have been applied in many fields, such as image registration [1-2], frame displacement for old films, and matching periodic DNA sequences [3], and so on. It has been known that if there is a strong similarity between two signals, the sharp peak appears in the POC functions. Many matching techniques using the POC functions are mainly based on this property. However, this property is known empirically and thus validity of the POC functions as a matching method has not been mathematically guaranteed.

In Refs. [4-5], the authors have analyzed the expectation and variance of the POC functions, assuming the phase spectrum differences to be random variables which are statistically constant for frequency indices. As a result, the authors have proved that the peak of the POC functions decreases and the variances of the POC functions increase as the variance of the phase spectrum differences increases. This result mathematically guarantees validity of the POC functions for evaluating similarity between two signals.

The aim and subject of this paper are the same as those of Refs. [4-5]. But, in this paper, we will first analyze properties of the POC functions with different assumptions of the phase spectrum differences from those in Refs. [4-5], that is, in case of statistically proportional phase differences to frequency indices. We will next discuss validity of the POC functions for evaluating similarity between two signals.

2 Some basic properties of phase-only correlation functions

Consider complex discrete-time signals \(x(n)\) and \(y(n)\) of length \(N\). The discrete Fourier transforms of \(x(n)\) and \(y(n)\) are given respectively as follows:

\[
X(k) = \sum_{n=0}^{N-1} x(n)W_N^{nk} = |X(k)|e^{j\theta_k}
\]

\[
Y(k) = \sum_{n=0}^{N-1} y(n)W_N^{nk} = |Y(k)|e^{j\phi_k}
\]

\((k = 0,1,2,\ldots,N-1)\)

where \(W_N = \exp(-j2\pi/N)\) is the twiddle factor, \(|X(k)|\) and \(|Y(k)|\) are the amplitude spectra and \(\theta_k\) and \(\phi_k\) are the phase spectra of \(x(n)\) and \(y(n)\), respectively. The POC function \(r(m)\) between two signals \(x(n)\) and \(y(n)\) is defined as follows:

\[
r(m) = \frac{1}{N} \sum_{k=0}^{N-1} \frac{X(k)Y^*(k)}{|X(k)||Y(k)|} W_N^{-mk}
\]

\((m = 0,1,2,\ldots,N-1)\)

where \(|X(k)||Y(k)| \neq 0\), the asterisk “\(^*\)” represents complex conjugate, and \(\alpha_k = \theta_k - \phi_k\) are the phase spectrum differences. The POC function \(r(m)\) between two signals \(x(n)\) and \(y(n)\) is described as the inverse discrete Fourier transform of the normalized cross-power spectrum as shown in Eq. (3). On the other hand, the POC function \(r(m)\) can be interpreted as the inverse discrete Fourier transform of the phase factor \(e^{j\alpha_k}\) as shown in Eq. (4). Thus, we can obtain the POC function \(r(m)\) of two signals, given their phase spectrum...
differences \( a_k \). It is generally complex for complex signals \( x(n) \) and \( y(n) \).

### 3 POC functions with proportional phase spectrum differences to frequency

When the phase spectra of two signals are completely equal, then their phase spectrum differences are \( \alpha_k = \theta_k - \varphi_k = 0 \), that is, \( \alpha_k \) are zero phase. In this case, the POC function \( r(m) \) is the delta function \( \delta(m) \) as easily shown below:

\[
r(m) = \frac{1}{N} \sum_{k=0}^{N-1} e^{j \omega k} W_N^{-mk} = \delta(m) = \begin{cases} 1, & m = 0 \\ 0, & m \neq 0 \end{cases} \quad (5)
\]

When the phase spectrum differences \( \alpha_k \) are proportional to frequency indices \( k \) for an integer \( \gamma \), that is, \( \alpha_k \) are linear phase as follows:

\[
\alpha_k = \begin{cases} -\gamma \left( \frac{2\pi}{N} \right) k, & k = 0, 1, 2, \ldots, [N/2] - 1 \\ \gamma \left( \frac{2\pi}{N} \right) (N-k), & k = [N/2], \ldots, N - 1, \end{cases} \quad (6)
\]

where \( [s] \) is the ceiling function of \( s \); that is, the smallest integer not less than \( s \), and then the POC function \( r(m) \) can be derived as follows:

\[
r(m) = \delta(m - \gamma). \quad (7)
\]

The above equation is easily seen to be equivalent to Eq. (5) for \( \gamma = 0 \).

Given two signals \( x(n) \) and \( y(n) = x(n - \gamma) \) with an integer \( \gamma \), which represents a time delay of \( y(n) \) to \( x(n) \), the phase spectrum differences \( \alpha_k \) are given by Eq.(6), and their POC function will be Eq.(7).0. Many matching techniques using the POC functions are mainly based on the fact that the POC functions become the delta function as Eq. (5) or Eq. (7) when the phase spectrum differences are zero (zero phase) or proportional to frequency indices (linear phase).

However, it is very rare that all the phase spectrum differences \( \alpha_k \) are zero or proportional to frequency indices \( k \) as shown in Eq. (6). Therefore, we have to consider the POC functions when all the phase spectrum differences \( \alpha_k \) are not zero or not proportional to frequency indices \( k \). Figure 1 shows an example of the POC function when all the phase spectrum differences \( \alpha_k \) are not zero. Then, the POC function is not a simple delta function, but its peak decreases from the unity and other values become nonzero.

![Figure 1 POC function when phase spectrum differences \( \alpha_k \) are nonzero and not proportional to frequency indices \( k \)](image)

### 4 Analysis of POC functions with statistically proportional phase spectrum differences to frequency indices

#### 4.1 Stochastic assumptions for phase spectrum differences

We assume the phase spectrum differences \( \alpha_k \) given by

\[
\alpha_k = \begin{cases} -\gamma_k \left( \frac{2\pi}{N} \right) k, & k = 0, 1, 2, \ldots, [N/2] - 1 \\ \gamma_k \left( \frac{2\pi}{N} \right) (N-k), & k = [N/2], \ldots, N - 1, \end{cases} \quad (8)
\]

where variables \( \gamma_k \) are Gaussian random variables with zero mean and variance \( \sigma^2 \), and statistically independent for frequency indices \( k \). Then, the phase spectrum differences \( \alpha_k \) are also Gaussian random variables with zero mean and variance as

\[
\begin{align*}
\left( \frac{2\pi}{N} \right)^2 \sigma^2 k^2, & \quad k = 0, 1, 2, \ldots, [N/2] - 1 \\
\left( \frac{2\pi}{N} \right)^2 \sigma^2 (N-k)^2, & \quad k = [N/2], \ldots, N - 1,
\end{align*}
\]

and are statistically independent for frequency indices \( k \). The above assumption for \( \alpha_k \) is a stochastic version of the assumption Eq.(6).

From the above assumption, we will derive statistical properties of the POC functions. We first need to derive statistical properties of the phase factor \( e^{j\alpha_k} \) in order to find statistical properties of the POC functions. We denote the expectation and variance of the phase factor \( e^{j\alpha_k} \) as

\[
A_k = E[e^{j\alpha_k} \] and \( V_k = Var[e^{j\alpha_k}] \, , \text{ where } E[\cdot] \text{ and } Var[\cdot] \text{ denote the expectation and variance operators, respectively.}
\]

The expectation \( A_k \) of the phase factor \( e^{j\alpha_k} \) can be derived as follows:

\[
A_k = \begin{cases} \beta^k, & \quad k = 0, 1, 2, \ldots, [N/2] - 1 \\
\beta (N-k)^2, & \quad k = [N/2], \ldots, N - 1 
\end{cases} \quad (10)
\]

where \( \beta \) is set to

\[
\beta = \begin{cases} 0, & \quad k = 0, 1, 2, \ldots, [N/2] - 1 \\
1, & \quad k = [N/2], \ldots, N - 1 
\end{cases}
\]
Proceedings of IC-NIDC2012

\[ \beta = \exp \left( -\frac{(2\pi/N)^2 \sigma^2}{2} \right). \]  \hspace{1cm} (11)

The inverse discrete Fourier transform of \( A_k \) is set to \( a_m \) for later use, i.e.,

\[ a_m = \frac{1}{N} \sum_{k=0}^{N-1} A_k W_N^{-mk}, m = 0, 1, 2, \ldots, N - 1. \]  \hspace{1cm} (12)

The variance \( V_k \) of the phase factor \( e^{i\alpha_k} \) can be derived as follows:

\[ V_k = \text{Var} \left[ e^{i\alpha_k} \right] = E \left[ e^{i\alpha_k} \right] - E \left[ e^{i\alpha_k} \right]^* \]

\[ = \begin{cases} 1 - \beta^{2k^2}, & k = 0, 1, 2, \ldots, [N/2] - 1 \\ 1 - \beta^{2(N-k)^2}, & k = [N/2], \ldots, N - 1. \end{cases} \]  \hspace{1cm} (13)

Since the phase spectrum differences \( \alpha_k \) are independent for frequency indices \( k \) by assumption, we have

\[ E \left[ e^{i\alpha_k} e^{i\alpha_q} \right] = \begin{cases} 1, & p = q \\ 0, & p \neq q. \end{cases} \]  \hspace{1cm} (14)

Figure 2 shows an example of the POC function with stochastic phase differences shown in Eq. (9), where we should note that under the assumption Eq. (9), two input signals \( x(n) \) and \( y(n) \) are generally complex and thus their POC function is also complex.

**Figure 2** Example of POC function

### 4.2 Statistical properties of POC functions

From Eqs. (10) to (14), we will derive statistical properties of the POC functions.

#### 4.2.1 Expectation of POC functions

From Eq. (10), the expectation \( E[r(m)] \) can be derived as follows:

\[ E[r(m)] = E \left[ \sum_{k=0}^{N-1} e^{i\alpha_k} W_N^{-mk} \right] \]

\[ = \frac{1}{N} \sum_{k=0}^{N-1} E \left[ e^{i\alpha_k} \right] W_N^{-mk} \]

\[ = \frac{1}{N} \sum_{k=0}^{N-1} A_k W_N^{-mk} \quad (= a_m) \]  \hspace{1cm} (15)

#### 4.2.2 Variance of POC functions

The variance \( \text{Var}[e^{i\alpha_k}] \) can be derived as follows:

\[ \text{Var}[r(m)] = E[r(m)r^*(m)] - E[r(m)]E[r^*(m)] \]

\[ = E[r(m)r^*(m)] - a_m a_m^*. \]  \hspace{1cm} (17)

From Eqs. (10), (13), and (14), the mean square \( E[r(m)r^*(m)] \) can be derived as follows:

\[ E[r(m)r^*(m)] = E \left[ \sum_{k=0}^{N-1} e^{i\alpha_k} W_N^{-mk} \right] \]

\[ = \frac{1}{N} \sum_{k=0}^{N-1} W_N^{-mk} E \left[ e^{i\alpha_k} e^{-i\alpha_k} \right] \]

\[ = \frac{1}{N} \sum_{k=0}^{N-1} W_N^{-mk} E \left[ e^{i\alpha_k} \right] E \left[ e^{-i\alpha_k} \right] \]

\[ = \frac{1}{N} \left( N - \sum_{k=0}^{N-1} W_N^{-mk} E \left[ e^{i\alpha_k} \right] E \left[ e^{-i\alpha_k} \right] \right) \]

\[ = \frac{1}{N} \left( N - \sum_{k=0}^{N-1} A_k A_k^* \right) \]

\[ = \frac{1}{N} \left( N - \sum_{k=0}^{N-1} A_k A_k^* \right) + a_m a_m^*. \]  \hspace{1cm} (18)

By substituting Eq. (18) into Eq. (17), the variance \( \text{Var}[r(m)] \) can be rewritten as follows:

\[ \text{Var}[r(m)] = \frac{1}{N} \left( N - \sum_{k=0}^{N-1} A_k A_k^* \right) \]

\[ = \frac{1}{N} \left( \frac{N-1}{N} A_k A_k^* \right) + a_m a_m^*. \]  \hspace{1cm} (19)

### 5 Analysis of POC functions under another assumption

In Section 4, we have considered the assumption given in Eq. (8). However, under this assumption, it is difficult to express the expectation Eq. (16) and variance Eq. (19) of POC functions in closed form. Therefore, we next consider the following assumption instead.
5.1 Stochastic assumption for phase spectrum differences

We replace the time index $k$ with $\sqrt{k}$ in Eq. (8), that is, we adopt the following assumption for the phase spectrum differences:

$$
\alpha_k = \begin{cases}
-2\gamma_k \sqrt{N}, & k = 0, 1, 2, \ldots, [N/2] - 1 \\
\gamma_k \sqrt{N - k}, & k = [N/2], \ldots, N - 1,
\end{cases}
$$

(20)

where $\gamma_k$ are also assumed to be Gaussian random variables with zero mean and variance $\sigma^2$, and statistically independent for frequency indices $k$. Thus, the phase spectrum differences $\alpha_k$ are Gaussian random variables with zero mean and variance

$$
\frac{2\pi^2}{N} \sigma^2, \quad k = 0, 1, 2, \ldots, [N/2] - 1
$$

$$
\frac{2\pi^2}{N} \sigma^2(N - k), \quad k = [N/2], \ldots, N - 1,
$$

(21)

and are statistically independent for frequency indices $k$.

Then, the expectation $A_k$ of the phase factor $e^{j\alpha_k}$ can be derived as follows:

$$
A_k = E[e^{j\alpha_k}] = \begin{cases}
\beta^k, & k = 0, 1, 2, \ldots, [N/2] - 1 \\
\beta^{(N-k)}, & k = [N/2], \ldots, N - 1.
\end{cases}
$$

(22)

The variance $V_k$ of the phase factor $e^{j\alpha_k}$ can be derived as follows:

$$
V_k = \text{Var}[e^{j\alpha_k}] = E[e^{j\alpha_k}(e^{j\alpha_k})^*] - E[e^{j\alpha_k}]E[(e^{j\alpha_k})^*] = \begin{cases}
1 - \beta^{2k}, & k = 0, 1, 2, \ldots, [N/2] - 1 \\
1 - \beta^{2(N-k)}, & k = [N/2], \ldots, N - 1.
\end{cases}
$$

(23)

Under the assumption in Eq. (20), we will derive the expectation and variance of the POC functions in closed form.

5.2 Statistical properties of POC functions

5.2.1 Expectation of POC functions

Replacing $\beta^k$ with $\beta^k$ and $\beta^{(N-k)}$ with $\beta^{(N-k)}$ in Eq.(16) yields a closed form expression of the expectation $E[r(m)]$ of the POC function as follows:

$$
E[r(m)] = \frac{1}{N} \left( \sum_{k=0}^{(N-1)/2} \beta^k W_{N}^{-mk} + \sum_{k=0}^{N-1} \beta^{(N-k)} W_{N}^{-mk} \right) = \begin{cases}
\frac{1}{N} \left( 1 + \beta^2(-1)^m + \sum_{k=1}^{N-1} \beta^k \cos \left( \frac{2\pi}{N} mk \right) \right) & (\text{for even } N) \\
\frac{1}{N} \left( 1 + \sum_{k=1}^{(N-1)/2} \beta^k \cos \left( \frac{2\pi}{N} mk \right) \right) & (\text{for odd } N)
\end{cases}
$$

$$
= \frac{1}{N} \left( \frac{\beta^2}{1 - \beta^2} + \frac{\beta^{(N/2)}(-1)^m}{1 - \beta^2} \right)
$$

(24)

Figure 3 shows an example of the expectation $E[r(m)]$ of length $N = 16$. The imaginary parts of the expectation $E[r(m)]$ are seen to be zero from Eq. (24) and are thus omitted in Figure 3. We should note from Figure 4 that the expectation $E[r(0)]$, which is the expectation of the peak of the POC function, is monotonically decreasing with respect to the variance $\sigma^2$.

5.2.2 Variance of POC functions

Replacing $\beta^k$ with $\beta^k$ and $\beta^{(N-k)}$ with $\beta^{(N-k)}$ yields a closed form expression of the variance $\text{Var}[r(m)]$ as follows:

$$
\text{Var}[r(m)] = \frac{1}{N} \left( 1 - \frac{1}{N} \sum_{k=0}^{(N-1)/2} \beta^2 k + \frac{N-1}{N} \beta^{2(N-k)} \right)
$$

$$
= \frac{1}{N} \left( 1 - \frac{1}{N} \left( 1 - \beta^2 + \beta^{2(N/2)} \right) \right)
$$

(25)
Figure 4 shows an example of the variance \( \text{Var}[r(m)] \) of the POC function with length \( N = 16 \). We should note from Figure 4 that the variance \( \text{Var}[r(m)] \) is monotonically increasing with respect to the variance \( \sigma^2 \).

6 Qualitative properties of POC functions

In this section, we discuss qualitative properties of the POC functions.

6.1 Properties of expectation of POC functions

It is easy to see from Eq. (16)

\[
|E[r(m)]| \leq E[r(0)] \leq 1
\]

\((m = 0, 1, 2, \ldots, N - 1)\).

The same inequality also holds in Eq. (24). The above equation shows that the expectation \( E[r(m)] \) has its maximum value at \( m = 0 \), where the peak of the POC functions appears.

We can easily observe from Eq. (16) and Eq. (24) that the expectation \( E[r(0)] \) (i.e., the peak of the POC functions) is monotonically increasing with respect to \( \beta \). Remembering that \( \beta \) in Eq. (11) is monotonically decreasing with respect to \( \sigma^2 \), to which the phase spectrum differences \( \alpha_k \) are proportional as shown in Eqs. (8) and (20), we can state that the expectation \( E[r(0)] \) monotonically decreases as the variance of the phase spectrum differences \( \alpha_k \) increases.

6.2 Properties of variance of POC functions

We can also observe that the variance \( \text{Var}[r(m)] \) in Eq. (19) and Eq. (25) is monotonically decreasing with respect to \( \beta \). Remembering again that \( \beta \) is monotonically decreasing with respect to \( \sigma^2 \), to which the phase spectrum differences \( \alpha_k \) are proportional, we can state that the variance \( \text{Var}[r(m)] \) monotonically increases as the variance of the phase spectrum differences \( \alpha_k \) increase.

7 Conclusions

In this paper, we have analyzed the expectation and variance of the POC functions, assuming the phase spectrum differences between two signals to be random variables and statistically proportional to frequency indices. This analysis leads to the qualitative property that the expectation \( E[r(0)] \), that is, the peak of POC functions, is monotonically decreasing and the variance of the POC functions is monotonically increasing with respect to the variance of the phase spectrum differences.

Generally speaking, the bigger the difference between two signals for the POC functions is, the larger the variance of their phase spectrum differences become. Thus, as increase of the difference between two signals for the POC functions, the expectation \( E[r(0)] \) of the POC functions decreases and the variance of the POC functions increases. This property guarantees validity of the POC functions as matching techniques.

Acknowledgement

A part of results presented in this paper was achieved by carrying out an MIC program “Research and development of technologies for realizing disaster-resilient networks” (the no. 3 supplementary budget in 2011 general account).

References