SUMMARY This paper proposes statistical analysis of phase-only correlation functions based on linear statistics and directional statistics. We derive the expectation and variance of the phase-only correlation functions assuming phase-spectrum differences of two input signals to be probability variables. We first assume linear probability distributions for the phase-spectrum differences. We next assume circular probability distributions for the phase-spectrum differences, considering phase-spectrum differences to be circular data. As a result, we can simply express the expectation and variance of phase-only correlation functions as linear and quadratic functions of circular variance of phase-spectrum differences, respectively.

Keywords: phase-only correlation functions, directional statistics, circular probability distributions, mean direction, circular variance, von-Mises distribution

1. Introduction

Phase-only correlation (POC) functions have been widely used for evaluating similarity between two signals. They have been applied in many fields, such as image registration [1]–[6], pattern recognition [7]–[9], motion estimation [10]–[12], the frame displacement for old films [13], [14], matching periodic DNA sequences [15], optics [16], and so on. It has been known that the POC function is the delta function with two equal phase spectra. Therefore, we have to clarify the difference between the phase spectra of two signals. In such cases, the property of the POC function is not completely equal. Therefore, we next propose statistical analysis method for POC functions with stochastic phase-spectrum differences based on directional statistics [19]. Directional statistics are statistics for circular, spherical, and toroidal data, such as wind directions, directions of migrating birds, arrival times of patients at a casualty unit in a hospital, and so on [20]–[23]. This strategy is suitable for statistical analysis for phase-spectrum differences since it deals with phase-spectrum differences as circular data. We assume phase-spectrum differences between two signals to be random variables with some linear probability distributions, such as Gaussian distribution. We derive general expressions of the expectation and variance of the POC functions with stochastic phase-spectrum differences [17], [18]. In addition to the expectation and variance of the POC functions presented in [17], [18], we give first- and second-order statistics of the POC functions completely in this paper. Under this assumption, phase-spectrum differences can take arbitrary real value in the range of \((-\infty, \infty)\). However, phase-spectrum differences should be real values in the range of \([-\pi, \pi]\) since they are angle data. We have to consider phase-spectrum differences as circular data, rather than linear data.

Therefore, we next propose statistical analysis method for POC functions with stochastic phase-spectrum differences based on directional statistics [19]. Directional statistics are statistics for circular, spherical, and toroidal data, such as wind directions, directions of migrating birds, arrival times of patients at a casualty unit in a hospital, and so on [20]–[23]. This strategy is suitable for statistical analysis for phase-spectrum differences since it deals with phase-spectrum differences as circular data. We assume phase-spectrum differences between two signals to be random variables with some circular probability distributions. We can simply express the expectation and variance of phase-only correlation functions as linear and quadratic functions of circular variance of phase-spectrum differences, respectively.

2. Phase-Only Correlation Functions

2.1 Definition

Consider complex discrete-time signals \(x(n)\) and \(y(n)\) of length \(N\). The discrete Fourier transforms of \(x(n)\) and \(y(n)\) are given by

\[
X(k) = \text{DFT}[x(n)] = \sum_{n=0}^{N-1} x(n)W_N^{|n-k|} = |X(k)|e^{j\theta_k} \quad (1)
\]

\[
Y(k) = \text{DFT}[y(n)] = \sum_{n=0}^{N-1} y(n)W_N^{|n-k|} = |Y(k)|e^{j\theta_k} \quad (2)
\]

respectively, where \(j\) is the imaginary unit, \(W_N = e^{2\pi j/N}\)

\[\begin{align*}
X(k) &= \sum_{n=0}^{N-1} x(n)W_N^{|n-k|} = |X(k)|e^{j\theta_k} \\
Y(k) &= \sum_{n=0}^{N-1} y(n)W_N^{|n-k|} = |Y(k)|e^{j\theta_k}
\end{align*}\]

\[W_N = e^{2\pi j/N}\]

where \(k = 0, 1, \cdots, N - 1\).

\[\begin{align*}
X(k) &= \sum_{n=0}^{N-1} x(n)W_N^{|n-k|} = |X(k)|e^{j\theta_k} \\
Y(k) &= \sum_{n=0}^{N-1} y(n)W_N^{|n-k|} = |Y(k)|e^{j\theta_k}
\end{align*}\]

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Y(k) &= \sum_{n=0}^{N-1} y(n)W_N^{|n-k|} = |Y(k)|e^{j\theta_k}
\end{align*}\]

\[W_N = e^{2\pi j/N}\]
exp(−j2π/N) is the twiddle factor, \( \theta_k \) and \( \phi_k \) are phase spectra of \( x(n) \) and \( y(n) \), respectively. The phase-only correlation (POC) function \( r(m) \) between two signals \( x(n) \) and \( y(n) \) is defined by the inverse discrete Fourier transform of normalized cross-power spectrum between two signals \( x(n) \) and \( y(n) \) as follows:

\[
r(m) = \text{IDFT} \left[ \frac{X(k)Y^*(k)}{|X(k)Y(k)|} \right] = \frac{1}{N} \sum_{k=0}^{N-1} e^{j\alpha_k} W^{-mk} \quad (m = 0, 1, \cdots, N-1)
\]

where \( \alpha_k = \theta_k - \phi_k \) are phase-spectrum differences. Note that the POC function \( r(m) \) can be considered as the inverse discrete Fourier transform of a signal \( e^{j\alpha_k} \), called phase factors.

### 2.2 Properties

Figure 1 shows examples for calculating POC functions for real signals with length \( N = 32 \). Figures 1(a), (b), and (c) illustrate the original signal \( x(n) \), its circular shifted version \( x_s(n) = x(n-\tau) \), where \( \tau = 3 \), \( n_N \) denotes \( n \mod N \), and noise-corrupted version \( x_n(n) = x(n) + \nu(n) \), where \( \nu(n) \) is an additive white Gaussian noise, respectively. Figures 1(d), (e), and (f) illustrate phase-spectrum differences between (d) \( x(n) \) and \( x_s(n) \), (e) \( x(n) \) and \( x_n(n) \), and (f) \( x_s(n) \) and \( x_n(n) \), respectively. Figures 1(g), (h), and (i) illustrate the POC functions between (g) \( x(n) \) and \( x(n) \), (h) \( x(n) \) and \( x_s(n) \), and (i) \( x(n) \) and \( x_n(n) \), respectively.

If two input signals are completely equal, phase spectra of two signals are equal, that is, \( \alpha_k = \theta_k - \phi_k = 0 \) as shown in Fig. 1(d). In this case, POC function \( r_{xx}(m) \) is the delta function \( \delta(m) \) since

\[
r_{xx}(m) = \frac{1}{N} \sum_{k=0}^{N-1} e^{j\alpha_k} W^{-mk} = \frac{1}{N} \sum_{k=0}^{N-1} W^{-mk} = \begin{cases} 1 & (m = 0) \\ 0 & (m \neq 0) \end{cases} = \delta(m).
\]
This result is shown in Fig. 1(g).

If two input signals are related by circular shift, that is, one is \( x(n) \) and the other is \( x_c(n) = x(n - \tau) \), phase spectra of two signals are linear with respect to frequency \( k \) as shown in Fig. 1(e). In this case, POC function \( r_{xx}(m) \) is the delta function shifted by \( \tau \) samples since

\[
r_{xx}(m) = \frac{1}{N} \sum_{k=0}^{N-1} W_N^{-\tau k} W_N^{-mk} = \frac{1}{N} \sum_{k=0}^{N-1} W_N^{(m + \tau)k} = \begin{cases} 1 & (m \text{ mod } N = -\tau) \\ 0 & (m \text{ mod } N \neq -\tau) \end{cases} = \delta(m + \tau)_N.
\]

Therefore, the location of the peak corresponds to the shift amount. This result is shown in Fig. 1(h).

These properties have been exploited in many matching techniques. However, in practical signal processing scene, it is quite unrealistic that the two input signals are equal or related by circular shift. The property in Eq. (4) holds only in case that phase spectra are completely equal, and the property in Eq. (5) holds only in case that two signals are related by circular shift.

In most realistic case, input signals are corrupted by noise, which causes corrupted phase spectra. As a result, POC function cannot be the delta function. Suppose that we have original signal \( x(n) \) and its noise-corrupted version \( x_c(n) = x(n) + \nu(n) \). As shown in Fig. 1(f), phase-spectrum differences between \( x(n) \) and \( x_c(n) \) seem to be a random sequence. The result of calculation of the POC function between these two signals is shown in Fig. 1(i). We can observe in this figure that \( |r(0)| \) decreases and \( |r(m \neq 0)| \) increase, compared with the delta function in Fig. 1(g). The additive noise causes such effects, which make wave form matching by POC functions difficult. Therefore, it is important to evaluate variation of the POC functions under the noise, assumed to be random variables.

Figure 2 shows simple numerical examples of the POC functions for stochastic phase-spectrum differences. We set length of signals to be \( N = 16 \). We assume phase-spectrum differences \( a_k \) follow Gaussian distribution \( N(0, \sigma^2) \), and calculate POC functions in Eq. (3) for \( \sigma^2 = 0, 0.25, 0.5, 1 \). We can observe that \( |r(0)| \) decreases as the variance \( \sigma^2 \) increases. On the other hand, \( |r(m \neq 0)| \) tend to increase as the variance \( \sigma^2 \) increases. In order to clarify the statistical properties of POC functions, we have to give some theoretical evidence for these experimental results of the POC functions with stochastic phase-spectrum differences. Behavior of peak value \( |r(0)| \) can be described by expectation of the POC function \( r(m) \). On the other hand, energy in sidelobe \( |r(m \neq 0)| \) can be described by variance of the POC function \( r(m) \). Therefore, in the next Section, we derive expectation and variance of POC functions with stochastic phase-spectrum differences.

3. Phase-Only Correlation Functions with Stochastic Phase-Spectrum Differences

In this section, we propose statistical analysis of phase-only correlation functions with stochastic phase-spectrum differences [17], [18]. We assume phase-spectrum differences as random variables following linear distributions.

3.1 Stochastic Assumptions for Phase-Spectrum Differences

Consider the following noise-corrupted signal \( x_c(n) \):

\[
x_c(n) = x(n) + \nu(n)
\]

where \( x(n) \) is an original signal and \( \nu(n) \) is an additive noise signal. We can express the noise-corrupted frequency spectrum \( X_c(k) \) by taking DFT of \( x_c(n) \) as follows:

\[
X_c(k) = \text{DFT}[x(n) + \nu(n)] = X(k) + V(k) = |X(k) + V(k)| e^{j\theta(k - \alpha_k)}
\]

where \( X(k) \) and \( V(k) \) are DFTs of \( x(n) \) and \( \nu(n) \), respectively. The magnitude spectrum \( |X(k)| \) and phase spectrum \( \theta(k) \) of the original signal \( x(n) \) are noise-corrupted by \( V(k) \) and \( -\alpha_k \), respectively. We can consider \( \alpha_k \) as phase-spectrum differences caused by the additive noise \( \nu(n) \). We can express the POC function between signals \( x(n) \) and \( x_c(n) \) as follows:

\[
r_{xx}(m) = \text{IDFT} \left[ \frac{X(k)X_c^*(k)}{|X(k)X_c(k)|} \right] = \text{IDFT} \left[ \frac{|X(k)|e^{j\theta(k)}|X(k) + V(k)|e^{-j\theta(k) - \alpha_k}}{|X(k)(X(k) + V(k))|} \right] = \text{IDFT} \left[ e^{j\alpha_k} \right]
\]

where noise-corrupted magnitude spectrum does not affect to the POC function. We consider \( \alpha_k \) to be random process as shown in Fig. 1(i). In order to derive the POC function given by Eq. (8), we assume phase-spectrum differences \( \alpha_k \).
as random variables with some probability distribution.

In order to analyse effects of stochastic phase-spectrum differences on the POC functions, we give some stochastic assumptions for phase-spectrum differences $\alpha_k$'s. We assume that phase-spectrum differences $\alpha_k$'s ($k = 0, 1, \cdots, N - 1$) are i.i.d. (independent, identically distributed) random variables. Since phase-spectrum differences $\alpha_k$'s are statistically independent for frequency indices $k = 0, 1, \cdots, N - 1$, we have

$$p(\alpha_k, \alpha_l) = p(\alpha_k)p(\alpha_l) \quad (k \neq l)$$

(9)

where $p(\alpha_k, \alpha_l)$ is the joint probability density function of $\alpha_k$ and $\alpha_l$, and $p(\alpha_k)$ and $p(\alpha_l)$ are their individual probability density functions. From Eq. (9), we have

$$E[\alpha_k\alpha_l] = E[\alpha_k]E[\alpha_l] \quad (k \neq l).$$

(10)

Similarly, for also phase factor $e^{j\alpha_k}$'s, we have

$$E[e^{j\alpha_k}e^{j\alpha_l}] = E[e^{j\alpha_k}]E[e^{j\alpha_l}] \quad (k \neq l).$$

(11)

For $k = l$, it is obvious that $E[e^{j\alpha_k}e^{j\alpha_l}] = 1$. In Eq. (3), phase-spectrum differences appear in form of phase factor $e^{j\alpha_k}$. Therefore, without loss of generality, we denote the expectation and variance of phase factor $e^{j\alpha_k}$ as $A = E[e^{j\alpha_k}]$ and $B = \text{Var}[e^{j\alpha_k}]$, where $E[\cdot]$ and $\text{Var}[\cdot]$ denote the expectation and variance operators, respectively. Since we have assumed phase-spectrum differences $\alpha_k$'s to be identically distributed, both of $A = E[e^{j\alpha_k}]$ and $B = \text{Var}[e^{j\alpha_k}]$ are constants not depending on frequency index $k$. These expressions are derived as

$$A = E[e^{j\alpha_k}]$$

(12)

$$B = \text{Var}[e^{j\alpha_k}]$$

$$= E[(e^{j\alpha_k} - E[e^{j\alpha_k}])^2]$$

$$= 1 - |A|^2.$$ 

(13)

The values $A$ and $B$ are determined by giving a probability density function of phase-spectrum differences $\alpha_k$'s. We can assume arbitrary probability density functions, such as Gaussian distribution, uniform distribution, Laplace distribution, Nakagami-Rice distribution, Rayleigh distribution, and so on.

### 3.2 General Expressions for Expectation and Variance of POC Function $r(m)$

We derive the expectation and variance of POC function $r(m)$. We first derive the expectation $E[r(m)]$ of POC function $r(m)$ in Eq. (3) as follows:

$$E[r(m)] = \left\{ \begin{array}{ll} A (m = 0) = A \delta(m) & \text{for } m \neq 0 \end{array} \right.$$ 

(14)

Equation (14) shows that $E[r(m)]$ has only non-zero value $A$ at $m = 0$. On the other hand, $E[r(m)] = 0$ for $m \neq 0$.

We next derive the variance $\text{Var}[r(m)]$ of POC function $r(m)$ as follows:

$$\text{Var}[r(m)] = E\left[(r(m) - E[r(m)])^2\right]$$

$$= E\left[(r(m))^2\right] - |A|^2 \delta(m).$$

(15)

From Eq. (15), it is necessary to derive the mean squared POC function $E[|r(m)|^2]$. We thus derive the mean squared POC function $E[|r(m)|^2]$ as follows:

$$E[|r(m)|^2] = \left\{ \begin{array}{ll} 1 & \text{for } k = l \end{array} \right.$$ 

(16)

From Eq. (11), $E[e^{j\alpha_k}e^{j\alpha_l}]$ can be expressed as follows:

$$E[e^{j\alpha_k}e^{j\alpha_l}] = \left\{ \begin{array}{ll} 1 & \text{for } k = l \end{array} \right.$$ 

(17)

Substituting Eq. (17) into Eq. (16) yields

$$E[|r(m)|^2] = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} |A|^2 W_{N}^{m(k-l)} + \frac{1}{N^2} \sum_{k=0}^{N-1} (1 - |A|^2) W_{N}^{m0}$$

$$= |A|^2 \left( \frac{1}{N} \sum_{k=0}^{N-1} W_{N}^{m} \right) + \frac{1}{N} (1 - |A|^2).$$

(18)

From Eq. (18) into Eq. (15), we have

$$\text{Var}[r(m)] = \frac{1}{N} (1 - |A|^2) = \frac{1}{N} B$$

(19)

$$= \left\{ \begin{array}{ll} A (m = 0) = A \delta(m) & \text{for } m \neq 0 \end{array} \right.$$ 

It is remarkable that $\text{Var}[r(m)]$ does not depend on index $m$, which means that the POC function $r(m)$ has constant variance for all $m$.

Equations (14) and (19) give the general expressions of the expectation and variance of POC function $r(m)$, respectively. Although it is sufficient to derive the expectation and variance of POC function for explaining the experimental results in Fig. 2, we give other first- and second-order statistics of the POC functions in Appendix.

### 3.3 Expressions of the Expectation and Variance of POC Functions Based on Characteristic Functions

Given a probability distribution for phase-spectrum differences $\alpha_k$'s, we can express the expectation and variance of
the POC functions by using characteristic function.

The characteristic function $\psi_{\alpha_k}(t)$ of a probability density function $p(\alpha_k)$ is defined by

$$\psi_{\alpha_k}(t) = \int_{-\infty}^{\infty} e^{it\alpha_k} p(\alpha_k) d\alpha_k.$$  \hspace{1cm} (20)

It is remarkable that $A = E[e^{i\beta_k}]$ in Eq. (12) can be expressed by characteristic function as follows:

$$A = E[e^{i\beta_k}] = \int_{-\infty}^{\infty} e^{i\alpha_k} p(\alpha_k) d\alpha_k = \psi_{\alpha_k}(1).$$ \hspace{1cm} (21)

Therefore, we can derive the expectation and variance of POC functions by using the characteristic functions.

Let the phase-spectrum differences $\alpha_k$’s be probability variables following Gaussian distribution $N(0, \sigma^2)$. Probability density function of $\alpha_k$ is given by

$$p(\alpha_k) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{\alpha_k^2}{2\sigma^2}} (-\infty < \alpha_k < \infty)$$ \hspace{1cm} (22)

of which characteristic function is known to be

$$\psi_{\alpha_k}(t) = e^{-\frac{1}{2}\sigma^2 t^2}.$$ \hspace{1cm} (23)

Therefore, we derive $A = E[e^{i\beta_k}]$ as follows:

$$A = \psi_{\alpha_k}(1) = e^{-\frac{1}{2}\sigma^2}.$$ \hspace{1cm} (24)

Substituting Eq. (24) into Eqs. (14) and (19), we have the expectation and variance of the POC function $r(m)$ as follows:

$$E[r(m)] = e^{-\frac{\sigma^2}{2}} \delta(m)$$ \hspace{1cm} (25)

As $\sigma^2$ increases, $E[r(0)]$ monotonically decreases from 1 to 0, and $\text{Var}[r(m)]$ monotonically increases from 0 to $1/N$, respectively. Figure 3 shows the expectation $E[r(0)]$ and variance $\text{Var}[r(m)]$ versus variance $\sigma^2$ with signals of length $N$. In Fig. 3, we set the range of the variance $\sigma^2$ to be $[0, 1]$ since the range of $\alpha_k$ can be approximately considered as $[-\pi, \pi]$ when $\sigma^2 = 1$. This result gives theoretical evidence for the experimental results shown in Fig. 2.

4. Statistical Analysis of POC Functions Based on Directional Statistics

In the previous section, we assumed phase-spectrum differences following linear distributions, such as Gaussian distribution. Under this assumption, phase-spectrum differences $\alpha_k$ can take arbitrary real value in the range of $(-\infty, \infty)$. However, phase-spectrum differences $\alpha_k$ should be real values in the range of $[-\pi, \pi]$ since they are angle data. In this section, we newly propose statistical approach for analysis of POC functions based on directional statistics [19]. We assume phase-spectrum differences as random variables following circular distributions.

4.1 Basis of Directional Statistics

Directional statistics are statistics for circular data, such as wind directions, directions of migrating birds, arrival times of patients at a casualty unit in a hospital, and so on [20]–[23]. Based on directional statistics, we have established a statistical analysis method for POC functions, which deal with phase-spectrum differences as circular data.

For circular probability variables $\alpha \in [-\pi, \pi]$, $p$-th trigonometric moment $A_p$ is defined by

$$A_p = E[e^{ip\alpha}].$$ \hspace{1cm} (27)

In terms of first-order trigonometric moment $A_1$, mean resultant length $\bar{R}$ and mean direction $\bar{\alpha}$ in $[-\pi, \pi]$ are defined by its absolute value and angle such as

$$\bar{R} = |A_1| (0 \leq \bar{R} \leq 1)$$ \hspace{1cm} (28)

$$\bar{\alpha} = \arg(A_1), \ (-\pi \leq \bar{\alpha} < \pi)$$ \hspace{1cm} (29)

Furthermore, circular variance $v$ is defined by

$$v = 1 - |A_1|. (0 \leq v \leq 1)$$ \hspace{1cm} (30)

A simple example for geometric interpretation of mean direction and circular variance of probability variables $\alpha_1$ and $\alpha_2$ is shown in Fig. 4. We have to note that mean direction $\bar{\alpha}$ and arithmetic mean $\bar{\alpha} = (\alpha_1 + \alpha_2)/2$ are not equal in general. From Fig. 4, we can note that mean resultant length $\bar{R}$ and circular variance $v$ can take real value in the range of $[0, 1]$ since first-order trigonometric moment $A_1 = E[e^{i\beta_k}]$ is always inside the unit circle on the complex plane.
4.2 Circular Probability Distributions

Probability distributions of circular random variables $\alpha$ are described by circular probability distribution functions $p(\alpha)$. The circular probability distribution function must satisfy the following conditions:

$$p(\alpha) \geq 0 \ ( -\pi \leq \alpha < \pi)$$  (31)

$$\int_{-\pi}^{\pi} p(\alpha) d\alpha = 1.$$  (32)

The characteristic function of a circular random variable $\alpha$ is defined by

$$\psi_{\alpha}(t) = \int_{-\pi}^{\pi} e^{i\alpha t} p(\alpha) d\alpha$$  (33)

which is essentially equivalent to Eq. (20) except for the interval of integration.

4.3 Description of POC Functions Based on Directional Statistics

We reveal relationship between directional statistics and POC functions. It is remarkable that expectation and variance of the POC functions in Eqs. (14) and (19) can be described in terms of circular variance of phase-spectrum differences.

First, the expectation of phase factor $A = E[e^{i\alpha}]$ given in Eq. (12) is found to be equal to first-order trigonometric moment $A_1$. Furthermore, relationship between first-order trigonometric moment and circular variance is given in Eq. (30). From Eqs. (14), (19), and (30), we derive the expectation and variance of the POC functions in terms of circular variance of phase-spectrum differences as follows:

$$|E[r(m)]| = |A| \delta(m)$$
$$= (1 - \nu) \delta(m)$$

$$\text{Var}[r(m)] = \frac{1}{N} (1 - |A|^2)$$
$$= \frac{1}{N} (1 - (1 - \nu)^2).$$  (34)  (35)

Therefore, expectation and variance of $r(m), E[r(m)]$ and $	ext{Var}[r(m)]$, are expressed as linear and quadratic functions of circular variance $\nu$, respectively. The value of circular variance $\nu$ is determined by giving a probability distribution of phase-spectrum differences $\alpha_k$.

From Eqs. (34) and (35), the expectation $|E[r(0)]|$ and variance $\text{Var}[r(m)]$ versus circular variance $\nu$ can be shown as Fig. 5. As circular variance $\nu$ increases from 0 to 1, expectation $|E[r(0)]|$ monotonically decreases from 1 to 0, and variance $\text{Var}[r(m)]$ monotonically increases from 0 to 1. These facts can be derived from Eqs. (34) and (35). These results give theoretical evidence for the experimental results as well as Fig. 3 in Sect. 3.

5. Numerical Example

Let the phase spectrum differences $\alpha_k$’s be probability variables following von-Mises distribution $VM(\tilde{\alpha}, \beta)$, where $\tilde{\alpha}$ is mean direction and $\beta$ is concentration. Von-Mises distribution, also called circular normal distribution, is well-known probability distribution in directional statistics. Probability density function of $\alpha_k$ is given by

$$p(\alpha_k) = \frac{1}{2\pi I_0(\beta)} e^{\beta \cos(\alpha_k - \tilde{\alpha})} \ ( -\pi \leq \alpha_k < \pi)$$  (36)
where $I_\nu(x)$ is the $\nu$th-order modified Bessel function of the first kind, defined by

$$I_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{l=0}^{\infty} \frac{1}{\Gamma(\nu+l+1)} \left(\frac{x}{2}\right)^{2l}. \quad (37)$$

The probability density function of von-Mises distribution $\text{VM}(0, \beta)$ is shown in Fig. 6. As concentration $\beta$ increases, von-Mises distribution becomes close to Gaussian distribution. On the other hand, substituting $\beta = 0$ in Eq. (36), we have

$$p(\alpha_k) = \frac{1}{2\pi} (-\pi \leq \alpha_k < \pi) \quad (38)$$

which yields Uniform distribution $U(-\pi, \pi)$.

The characteristic function of $\alpha_k$ following $\text{VM}(\bar{\alpha}, \beta)$ is given by

$$\psi_{\alpha_k}(t) = \frac{I_\nu(\beta)}{I_0(\beta)} e^{i\beta t}. \quad (39)$$

Therefore, we derive $A = E[e^{i\nu \alpha_k}]$ as follows:

$$A = \psi_{\alpha_k}(1) = \frac{I_1(\beta)}{I_0(\beta)} e^{i\beta}. \quad (40)$$

When mean direction $\bar{\alpha} = 0$, the above equation is rewritten as follows:

$$A = \psi_{\alpha_k}(1) = \frac{I_1(\beta)}{I_0(\beta)}. \quad (41)$$

On the other hand, circular variance of $\alpha_k$ is given by

$$v = 1 - \frac{I_1(\beta)}{I_0(\beta)} = 1 - |A| \quad (42)$$

which is equal to Eq. (30). Substituting Eq. (41) into Eqs. (14) and (19), we have the expectation and variance of the POC function $r(m)$ in terms of concentration $\beta$ as follows:

$$|E[r(m)]| = |A| \delta(m)$$

and

$$\text{Var}[r(m)] = \frac{I_1(\beta)}{I_0(\beta)} \delta(m). \quad (43)$$

The expectation $|E[r(0)]|$ and variance $\text{Var}[r(m)]$ of POC function $r(m)$ versus concentration $\beta$.

Figure 7 shows the expectation $|E[r(0)]|$ in Eq. (43) and variance $\text{Var}[r(m)]$ in Eq. (44) versus concentration $\beta = [0, 10]$ with signals of length $N$. We can see from Fig. 7 that as concentration $\beta$ increases, $|E[r(0)]|$ monotonically decreases from 1 to 0, and $\text{Var}[r(m)]$ monotonically increases from 0 to $1/N$, respectively.

6. Concluding Remarks

In this paper, we have proposed a statistical analysis method for the POC functions with stochastic phase-spectrum differences based on linear statistics and directional statistics. We first assumed phase-spectrum differences between two signals to be random variables with linear probability distributions and derived general expressions of the expectation and variance of the POC functions with stochastic phase-spectrum differences. Next, we assumed phase-spectrum differences between two signals to be random variables following a circular probability distribution. As a result, we can simply express the expectation and variance of phase-only correlation functions as linear and quadratic functions of circular variance of phase-spectrum differences, respectively. As circular variance of phase-spectrum differences between two signals increases, the expectation $|E[r(0)]|$ decreases and the variance $\text{Var}[r(m)]$ increases, respectively. This result mathematically guarantees the validity of the
POC functions used for similarity measure in matching techniques. In this paper, we assume that two signals are complex signals.

As our future works, we first analyse the properties of POC functions for real signals. Furthermore, assuming wrapped distributions seems to be useful for phase-spectrum differences, since we can treat linear distributions as if circular distributions. It is also our significant future work.

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References


Appendix: First- and Second-Order Statistics of Phase-Only Correlation Functions

First, we denote the POC function \( r(m) \) as follows:

\[
r(m) = r_r(m) + jr_i(m)
\]  

where \( r_r(m) \) and \( r_i(m) \) are real and imaginary parts of the POC function \( r(m) \), respectively. In order to completely describe first- and second-order statistics of the POC function \( r(m) \), it is necessary to derive \( E[r_r(m)], E[r_i(m)], E[r_r^2(m)], E[r_i^2(m)], \) and \( E[r_r(m)r_i(m)] \).

A.1 First-Order Statistics

It is obvious that the first-order statistics \( E[r_r(m)] \) and \( E[r_i(m)] \) are derived as

\[
E[r_r(m)] = \text{Re} [E[r(m)]] = \text{Re}[A] \delta(m)
\]

\[
E[r_i(m)] = \text{Im} [E[r(m)]] = \text{Im}[A] \delta(m)
\]

respectively.

A.2 Second-Order Statistics

In order to derive second-order statistics \( E[r_r^2(m)], E[r_i^2(m)], \) and \( E[r_r(m)r_i(m)] \), it is efficient to derive \( E[r^2(m)] \). In deriving the variance of the POC function \( r(m) \), we have considered

\[
|r(m)|^2 = r_r^2(m) + r_i^2(m).
\]

(4)

On the other hand,

\[
r^2(m) = r_r^2(m) - r_i^2(m) + 2jr_r(m)r_i(m).
\]

(5)

By taking expectations of Eqs. (4) and (5), we have

\[
E[|r(m)|^2] = E[r_r^2(m)] + E[r_i^2(m)]
\]  

(6)
\[ E[r^2(m)] = E[r_i^2(m)] - E[r_i^2(m)] + 2jE[r_i(m)r_i(m)]. \]  
(A.7)

From Eqs. (A.6) and (A.7), we can derive second-order statistics as follows:

\[ E[r_i^2(m)] = \frac{1}{2} \left[ E[r(m)^2] + \text{Re} \left[ E[r^2(m)] \right] \right] \]

\[ E[r_i^2(m)] = \frac{1}{2} \left[ E[r(m)^2] - \text{Re} \left[ E[r^2(m)] \right] \right] \]

\[ E[r_i(m)r_i(m)] = \frac{1}{2} \text{Im} \left[ E[r^2(m)] \right]. \]  
(A.8)

Since we have already derived \( E[r(m)^2] \) in Eq. (18), we next derive \( E[r_i^2(m)] \) as follows:

\[ E[r_i^2(m)] = E \left[ \frac{1}{N} \sum_{k=0}^{N-1} e^{j\alpha_k} W_N^{-mk} \right]^2 \]

\[ = E \left[ \frac{1}{N} \sum_{k=0}^{N-1} e^{j\alpha_k} W_N^{-mk} \right] \left( \frac{1}{N} \sum_{l=0}^{N-1} e^{j\beta_l} W_N^{-ml} \right) \]

\[ = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} E[e^{j\alpha_k} e^{j\beta_l}] W_N^{-m(k+l)} \]

\[ = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \sum_{i=1}^{N-1} E[e^{j\alpha_k} e^{j\beta_l}] W_N^{-m(k+l)}. \]  
(A.9)

From the independency of \( \alpha_k \)'s in Eq. (9), \( E[e^{j\alpha_k} e^{j\beta_l}] \) can be expressed as follows:

\[ E[e^{j\alpha_k} e^{j\beta_l}] = \begin{cases} E[e^{j2\alpha_k}] = A_2 & (k = l) \\ E[e^{j\alpha_k}]E[e^{j\beta_l}] = A_1^2 & (k \neq l). \end{cases} \]  
(A.10)

Substituting Eq. (A.10) into Eq. (A.9) yields

\[ E[r_i^2(m)] = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} A_2^2 W_N^{-m(k+l)} + \frac{1}{N^2} \sum_{k=0}^{N-1} (A_2 - A_1^2) W_N^{-2mk} \]

\[ = A_1^2 \left( \frac{1}{N} \sum_{k=0}^{N-1} W_N^{-mk} \right) \left( \frac{1}{N} \sum_{l=0}^{N-1} W_N^{-ml} \right) \]

\[ + \frac{1}{N^2} \sum_{k=0}^{N-1} (A_2 - A_1^2) \sum_{l=0}^{N-1} W_N^{-2mk} \]

\[ = A_1^2 \delta(m) + \frac{1}{N} (A_2 - A_1^2) \left( \delta(m) + \delta(m - N/2) \right) \]

\[ = \begin{cases} A_1^2 + \frac{1}{N} (A_2 - A_1^2) & (m = 0) \\ \frac{1}{N} (A_2 - A_1^2) & (m = N/2) \\ 0. \end{cases} \]  
(A.11)

Finally, \( E[r(m)^2] \) in Eq. (18) and \( E[r_i^2(m)] \) in Eq. (A.11), second-order statistics \( E[r_i^2(m)], E[r_i^2(m)], \) and \( E[r_i(m)r_i(m)] \) are derived as follows:

\[ E[r_i^2(m)] = \begin{cases} A_1^2 + \frac{1}{2N} (1 - 2A_1^2 + A_2) & (m = 0) \\ \frac{1}{2N} (1 - 2A_1^2 + A_2) & (m = N/2) \\ \frac{1}{2N} (1 - A_1^2 - A_2^2) \end{cases} \]  
(otherswise)  
(A.12)

\[ E[r_i^2(m)] = \begin{cases} A_1^2 + \frac{1}{2N} (1 - 2A_1^2 - A_2) & (m = 0) \\ \frac{1}{2N} (1 - 2A_1^2 - A_2) & (m = N/2) \\ \frac{1}{2N} (1 - A_1^2 - A_2^2) \end{cases} \]  
(otherswise)  
(A.13)

\[ E[r_i(m)r_i(m)] = \begin{cases} A_1A_1i + \frac{1}{2N} (A_2i - 2A_1A_1i) & (m = 0) \\ \frac{1}{2N} (A_2i - 2A_1A_1i) & (m = N/2) \\ 0 \end{cases} \]  
(otherswise)  
(A.14)

where \( A_1 = \text{Re}[A_1], A_1i = \text{Im}[A_1], A_2 = \text{Re}[A_2], \) and \( A_2i = \text{Im}[A_2], \) respectively.

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