Abstract There has been a constant interest in the design and implementation of digital filters with variable characteristics for different applications. In this article an attempt is made to review and systematize all known structures and methods of design of such filters. First the basic theory of the variable digital filters is introduced. Then FIR and IIR realizations with real and complex coefficients are considered, the known results for multidimensional variable filters are discussed and finally, typical implementations are overviewed. Recommendations for applications in different situations are given and unresolved problems are pointed out. This work is basically a review, but some of our original results are also included.

Keywords: Variable/tunable digital filters, FIR and IIR filters, complex coefficients, 1-D and 2-D digital filters

1. Introduction

In many signal processing applications a clear need to change the parameters of the filters used exists. Such applications are found in telecommunications, digital audio equipment, medical electronics, radar, sonar and control systems, adaptive and tracking systems, spectrum and vibration analyses, formant speech synthesizers and in numerous laboratory instruments. The most general term for filters with changeable parameters is “variable filters”, but they are often called “tunable” (although this term is correct only when frequency-related parameters are the subject of change), “adjustable” (this term is correct only for changes in some narrow range of values of a given parameter) or “programmable” (when parameters can be reprogrammed or are controlled by a computer). In analog variable filters, tuning is often achieved by trimming some passive element and thus the term “trimmed” is also used to refer to the filter.

There are many limitations and technical problems in realizing passive variable filters. The realization is easier with active filters (especially for filters with programmable parameters), but it becomes an easy task with practically unlimited possibilities in the case of digital variable filters.

In the most general case of a variable filter, all filter magnitude parameters (Fig. 1) might be subject to changes. Variable magnitude deviations \( \delta_s \) (in the stopband) and \( \delta_p \) (in the passband) are, however, not easily achievable – complicated recalculations of the entire transfer function are required and usually many circuit elements must be changed in order to obtain the new \( \delta_s \) and \( \delta_p \).

Fortunately, more often, only frequency parameters – cutoff frequency \( \omega_0 \) and stopband edge \( \omega_s \) of lowpass (LP) and highpass (HP) filters (Fig.1a) or \( \omega_{s1}, \omega_{s2}, \omega_{s3} \),

\[ M_{\omega_s}(\delta_s, \omega_{s1}, \omega_{s2}, \omega_{s3}) \]

and stopband (BP) or bandstop (BS) filters – are tuned. Tuning is often achieved by trimming some passive element and thus the term “trimmed” is also used to refer to the filter.

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Fig. 1 Illustration of the possible variable magnitude response parameters for (a) low-pass filter; (b) band-pass filter.

\( \omega_{s2} \), center frequency \( \omega_0 \) and bandwidth \( BW \) (Fig. 1b) of bandpass (BP) and bandstop (BS) filters – are tuned. This is an easier task compared with all-magnitude-parameters tuning and it can be further simplified, for example, by keeping \( BW \) constant and tuning only \( \omega_0 \), or by tuning \( BW \) and \( \omega_0 \) independently.

The rare case in which the entire magnitude response is varied, is the magnitude equalizer. The magnitude response in this case, however, is easily decomposed into a product of first- and second-order components described by simple frequency parameters such as \( \omega_i \) and \( BW \), which are usually tuned without problems. Thus any arbitrary magnitude response could be obtained by varying the frequency parameters \( \omega_i \) and \( BW \).

It should be noted that the magnitude responses of magnitude equalizers change gradually (no sharp jumps) compared to most other filters and thus they have lower values of the transfer function quality factors. This facilitates the implementation and reduces the problems of tuning and of stability control during the tuning process.

Finally, there is another group of variable circuits with constant magnitude, where only the phase response (group delay time) is varied. These are called
phase equalizers, but they will not be discussed in this paper.

The first publications concerning variable digital filters date back to the early 70’s [1]-[5] and the problems were already more generally treated in the early 80’s [6]-[9]. Nowadays there are already hundreds of publications and there is a need to systematize the collected knowledge – the methods proposed and structures developed. This is the first aim of the present work. Then, we try to compare all the methods and structures developed. This is the first aim of the present work. Then, we try to compare all the methods and structures developed. This is the first aim of the present work.

The spectral transformations of Constantinides [2] are based on the substitution

\[ z^{-1} \rightarrow T(z). \]  

Thus, starting from a LP prototype filter with a cutoff frequency \( \omega_c \) and transfer function \( H_p(z) \),

\[ H_p(z) = \sum_{i=0}^{M} a_i z^{-i} = \frac{1}{1 + \sum_{j=0}^{N} b_j z^{-j}}, \]  

(1)

and applying (1), it is possible to obtain a LP, HP, BP or BS filter with a new transfer function

\[ H(z) = H_p(c_d z^{-\beta}) \]  

and variable cutoff frequencies. For LP to LP and LP to HP transformations, \( T(z) \) is

\[ T(z) = \frac{z^{-1} + \beta}{1 + \beta z^{-1}}; \]

\[ \beta_{LP} = \frac{\sin[(\omega_c - \omega_c)/2]}{\sin[(\omega_c + \omega_c)/2]}; \]

\[ \beta_{HP} = \frac{-\cos[(\omega_c - \omega_c)/2]}{\cos[(\omega_c + \omega_c)/2]} \]

(3)

where \( \beta_{LP} = \cos \omega_0 = \cos [(\omega_{c1} + \omega_{c2})/2], \)

\[ k = \cot \frac{\omega_c - \omega_{c1}}{2} \]

and

\[ T(z) = z^{-2} - \frac{2\beta/(k + 1)}{1 - \beta z^{-1}} \]

(4)

The transformation of Oppenheim et al. [3] starts from a zero-phase FIR filter with transfer function \( H_0(z) \) and symmetric impulse response \( h_0(n) = h_0(-n) \). Any linear-phase FIR filter with transfer function \( H(z) \) and impulse response \( h(n) \) of length \( 2N+1 \) can be expressed through \( H_0(z) \) and \( h_0(n) \) [3]:

\[ H(z) = z^{-N} H_0(z); \]

\[ h(n) = h_0(n - N); \]

where the coefficients \( a(n) \) are related to \( h_0(n) \) through the coefficients of Chebyshev’s polynomials \( C_n(x) \) of the \( n \)-th order.

The frequency responses corresponding to (7) are

\[ H(e^{j\Omega}) = \sum_{n=0}^{N} a(n)(\cos \omega)^n, \]

(5)

The transformation of Oppenheim et al. [3] is based on the substitution of variables

\[ \cos \omega = \sum_{k=0}^{p} A_k (\cos \Omega)^k, \]

(6)

where \( \Omega \) is the frequency scale of the transformed filter. If \( Z = e^{j\Omega} \) is the \( z \)-variable corresponding to \( \Omega \), the new transfer functions \( H_d(Z) \) and \( H(Z) \) will also have linear phases and symmetric impulse responses like those of the starting transfer functions (7). The first relation in (7), however, changes to

\[ H(Z) = Z^{-N} H_0(Z); \]

and the length of the new filter becomes \( 2NP+1 \). The frequency response of this filter, on the other hand, may
vary with variations of coefficients \( A_i \) (9) and it could be used for tuning the new filter.

For the first-order transformation (9) \((P = 1)\),
\[
\cos \omega = A_0 + A_\cos \Omega ,
\]
the length of the new filter will be the same \((2N+1)\) and the two coefficients \(A_0\) and \(A_1\) will control the frequency response.

An additional restriction, \(A_0+A_1=1\), which makes the DC gain invariant, reduces the transformation to a single parameter one:
\[
\cos \omega = A_0 + (1 - A_0) \cos \Omega .
\]

If the prototype filter is LP with cutoff frequency \(\omega_0\), the transformed filter will have a cutoff frequency [3],
\[
\Omega_c = \cos^{-1} \frac{\cos \omega_0 - A_0}{1 - A_0},
\]
which is easily controlled by changing \(A_0\).

In \(z\)-variable transformation (9) becomes
\[
\frac{z + z^{-1}}{2} = \sum_{k=1}^{P} A_k \left( \frac{z + z^{-1}}{2} \right)^k .
\]

### 3. Variable FIR Digital Filters

3.1. Filters based on Oppenheim et al.’s transformation

Transformation (9), (13) is developed for the design of variable cutoff frequency linear-phase FIR digital filters and it is applied to Taylor’s structure [3] (Fig. 2a).

For first-order transformation, (13) changes to
\[
\frac{z + z^{-1}}{2} = A_0 + A_1 \frac{z + z^{-1}}{2},
\]
and it produces the structure shown in Fig. 2b. Its cutoff frequency can easily be changed by changing \(A_0\) and \(A_1\). Another approach is to decompose Taylor’s structure into a cascade connection of fourth-order sections and to apply transformation (14) to each of them.

It is shown in [4] that Oppenheim et al.’s transformation is good only for the range of values of \(d\Omega/d\omega\) (calculated from (10)) within the limits \(1 \leq d\Omega/d\omega \leq 2\).

For a higher-order transformation (13), the “0.5(1+Z)”-branches in Fig. 2a will be replaced by circuits of order \(P\) and the “\(z^{-1}\)”-branches by “Z”-branches [3]. Thus the order of the final filter increases \(P\) times.

The case of a single parameter (11) was investigated in [7] and it was found that the cutoff slope of the variable filter decreases when the cutoff frequency increases during tuning. Thus the frequency range over which the filter is tuned, may be considerably narrowed. This drawback, together with another one – having \(\Omega_c \geq \omega_0\) – could be avoided if second-order transformation (9), (13) is used:
\[
\cos \omega = A_0 + A_\cos \Omega + A_2 \cos^2 \Omega .
\]

The conditions \(A_1 = 1\) and \(A_0 = A_2\) are often introduced in order to preserve the number of variable parameters and the volume of computations.

In all cases up to now, only LP prototypes and LP variable filters have been considered. In [11] this approach is extended to linear-phase BP prototypes and linear-phase BP variable filters. If \(\omega_0\) and \(\omega_2\) are the cutoff frequencies of the prototype filter (Fig. 1b) and \(\Omega_1\) and \(\Omega_2\) are these of the transformed filter, it is easy to derive, from (10),
\[
A_i = \cos \omega_{i-1} - A_\cos \Omega_{i+1},
\]
\(i = 1\) or \(2\),
\[
A_i = (\cos \omega_{i-1} - \cos \omega_{i+2}) / (\cos \Omega_{i-1} - \cos \Omega_{i+2}).
\]

Thus, it becomes possible to tune the center frequency or the bandwidth by varying only the parameter \(A_i\). It is generally not possible to change the center frequency while keeping the bandwidth constant, but some ways of achieving this are suggested in [11].

Another approach to obtaining tunable BP filters from LP (not from BP as in [11]) prototypes, was given in [12], based on which transformation (10) is modified to
\[
\sin (\omega/2) = A_0 + A_\cos \Omega
\]
and the cutoff frequency \(\pm\omega_0\) of the LP prototype is mapped to \(\pm\Omega_1\) and \(\pm\Omega_2\). Then \(\Omega_1\) and \(\Omega_2\) are controlled by changing \(A_0\) and \(A_1\). Variable LP and HP filters, considered as special cases of BP filter, can also be designed using this approach. The prototype circuit, used in [12] has the same Taylor structure as in Fig. 2a.

Relations (16) and (17) are used in [13] (where (14) is called the Mobius transformation) as a starting point in the development of variable cutoff frequency linear-phase filters that are not based on the Taylor structure. The final result, however, is not a FIR filter.

3.2. Direct-form FIR variable filters

All known variable filters, based on Oppenheim et al.’s transformation, are realized as transformed Taylor’s structures (Fig. 2). The final circuit is quite specific, often not desirable, and, for higher-order transformation (13), too complicated. Neither is there a
simple relation between the filter coefficients and the cutoff frequency, which makes the hardware implementations difficult and impractical.

Another approach for producing direct-form linear-phase FIR variable filters with a simple relationship between filter coefficients and the cutoff frequency, was advanced by Jarske et al. [14], [15]. Starting from the impulse response $h_d(n)$ of an ideal LP filter, Jarske et al. applied the following approximation for the impulse response $h_0(n)$ of a LP filter with equal ripples $\delta_0$ and $\delta_1$ (Fig. 1), used as a prototype:

$$h_0(n) = \begin{cases} \frac{c(n)\omega_0 + d}{c(n)\sin (n\omega_0)} & \text{for } n = 0 \\ \frac{\sin (n\omega_0)}{\sin (n - 0.5)\omega_0} & \text{for } 1 \leq |n| \leq N \end{cases}$$  \quad (19)

where $\omega_0$ is the variable 6 dB cutoff frequency, $c(n)$ is a linear function of the cutoff frequency, and all $h_0(n)$ coefficients are also simple functions of $\omega_0$. $h_0(n)$ might be obtained using either a window-based $(h_0(n)=w(n)h_d(n))$, where $w(n)$ could be any symmetric window) or optimal (using the Remez exchange algorithm) method of design. When optimal designs are used, however, for filters of even-length $2N$, a better approximation is [15]

$$h_0(n) = c(n)\sin [\omega_0(n-0.5)],$$  \quad (20)

where $-N+1 \leq n \leq N$ and $c(n)$ is a constant. Coefficients $c(n)$ are determined in the design procedure and is used to adjust HP and BS filter responses ($d=0$ for an ideal LP filter).

It appears from (19) that the center coefficient $h_0(0)$ is a linear function of the cutoff frequency, and that all $h_0(n)$ coefficients are also simple functions of $\omega_0$. $h_0(n)$ might be obtained using either a window-based $(h_0(n)=w(n)h_d(n))$, where $w(n)$ could be any symmetric window) or optimal (using the Remez exchange algorithm) method of design. When optimal designs are used, however, for filters of even-length $2N$, a better approximation is [15]

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Another detail in the case of optimal design is that first an optimal LP filter $h_0(n)$ with 6 dB cutoff frequency $\omega_0$ is designed with $\omega_0$ chosen in such a way as to ensure that all coefficients $h_0(n)$ are nonzero. Jarske et al. [15] proposed

$$\begin{align*}
\omega_0 &= 2\pi[0.25+0.5(2N+1)] & \text{for odd-length } 2N+1 \\
\omega_0 &= 0.5\pi & \text{for even-length } 2N.
\end{align*}$$  \quad (21)

Then, coefficients $c(n)$ for (19) and (20) are calculated as

$$c(n) = \begin{cases} \frac{h_0(n)}{\sin (n\omega_0)} & \text{for } n \neq 0 \text{ and odd length} \\ \frac{h_0(n)}{\sin (n-0.5)\omega_0} & \text{for even length}, \end{cases}$$  \quad (22)

where $c(0)=1/\pi$ for odd length.

For a LP prototype with odd-length $2N+1$, a zero-phase HP filter can be obtained as its amplitude complementary. This filter will have the same cutoff frequency $\omega_0$ as that of the LP filter (which is easily made variable), and impulse response [15]

$$h_{HP}(n) = \begin{cases} 1-h_0(n) & \text{for } n = 0 \\ -h_0(n) & \text{for } 1 \leq |n| \leq N. \end{cases}$$  \quad (23)

The BP filter can be obtained by using a modulation scheme, and it will have impulse response

$$h_{BPV}(n) = 2\cos (\omega_0 n) h_0(n),$$  \quad (24)

where $\omega_0$ is the desired center frequency (Fig.1b) introduced by the modulation scheme.

The BS filter is easily designed by combining (23) and (24), while a notch filter can be obtained by taking the difference between the prototype LP filter and its complementary filter. The impulse response of such a notch filter is [15]

$$h_{BS}(n) = \begin{cases} 2h_0(n)-1 & \text{for } n = 0 \\ 2h_0(n) & \text{for } 1 \leq |n| \leq N. \end{cases}$$  \quad (25)

For even-length $2N$, expressions (23) and (24) change to

$$h_{BPV}(n) = \cos (\omega_0 n) h_0(n) \quad 0 < n < 2N$$  \quad (26)

$$h_{BPV}(n) = 2\cos (\omega_0 n) h_0(n) \quad -N+1 < n < N$$  \quad (27)

and the HP filter in this case is not the complementary one of the LP filter.

This method was extended by Jarske et al. [14], [15] to the design of variable filters with $\delta_0 \neq \delta_1$.

Although filter coefficients $h(n)$ are a simple function of the cutoff frequency (19), (20), this function (sine-function) is difficult to calculate in real-time implementations. Jarske et al. suggested the use of pre-calculated look-up tables, series expansions of the sine-function or recursive calculation with a digital sine-wave generator [15]. They also showed [16] how a LP variable filter can be implemented with a signal processor and a 1024-word look-up table containing one whole cycle of the sine function. A 58th-order variable center frequency BP filter is also described in [16]. In this case the look-up table contains the cosine function (see (24)). If a digital sine-wave generator is used, since this generator is based on a second-order recursive digital filter, Jarske et al. [14]-[16] proposed to realize this filter in a coupled form, thereby simultaneously providing sine and cosine functions.

There are some other drawbacks concerning the method of Jarske et al. First, it is not based on any clear and strong background, and some functions (for example, (21)) and selections are results of experiments collected after designing a great number of filters. Also, the passband and stopband edges cannot be set to the desired frequencies and the deviations $\delta_0$ and $\delta_1$ cannot be predicted in the design process when $\delta_0 \neq \delta_1$ [17]. All these disadvantages are avoided in a method developed by Yoshida et al. [17]. The principle of this method is based on the fact that the cross-sectional characteristics of a two-dimensional (2-D) filter along a line, vary if the intersection of this line is changed. This can be easily illustrated by a quadrantly symmetric FIR 2-D
filter with a hexagonal passband. Let the transfer function of this FIR filter be

\[ H(z_1, z_2) = \sum_{n_1=-N_1-1}^{N_1-1} \sum_{n_2=-N_2-1}^{N_2-1} h(n_1, n_2) z_1^{-n_1} z_2^{-n_2}, \tag{28} \]

where

\[ h(i,j) = h(-i,-j) = h(-i,j) \tag{29} \]

is its impulse response and \( N_1 \) and \( N_2 \) are both odd numbers (modifications are necessary for an even-tap filter). After setting \( z_1 = e^{j\omega_1}, z_2 = e^{j\omega_2}, \omega_1 = \omega \) and \( \omega_2 = 2\pi k \), the two-dimensional frequency response (28) changes to [17]

\[ H[e^{j\omega_1}] = h(0,0) + 2 \sum_{n_1=1}^{N_1-1} h(0,n_2) \cos(2\pi n_2 k) + 2 \sum_{n_2=1}^{N_2-1} g(n_1) \cos(2\pi n_1 \omega_2) \tag{30} \]

which is a frequency response of a 1-D linear-phase FIR filter with the impulse response

\[ g(n_1) = h(n_1,0) + 2 \sum_{n_2=1}^{N_2-1} h(n_1,n_2) \cos(2\pi n_2 \omega_2) \tag{31} \]

It is clear that the frequency response \( H[e^{j\omega_1}] \) and the impulse response \( g(n_1) \) will both vary with \( k \).

The design procedure developed in [17] includes the design of a 2-D hexagonal FIR filter in such a way that the resulting variable 1-D filter would satisfy the prescribed specifications. The calculated coefficients \( h(i,j) \) are stored and then the coefficients \( g(n_1) \) are calculated from the stored \( h(i,j) \) for a given parameter \( k \). The transition band of the real 2-D filter is taken into account. This approach can be used to design variable HP and BP filters corresponding to 2-D HP and BP prototypes.

The volume of calculations is obviously higher than that in Jarske et al.’s method, so this approach is recommended for applications where tuning (updating of the filter coefficients) is required only occasionally and the requirements for precise tuning are higher.

An interesting way to change the cutoff frequency of a linear-phase FIR filter is suggested in Ref. [18]. Starting from a filter with cutoff frequency \( \omega_0 \), picking up every \( M \)-th value of the impulse response (using every \( M \)-th tap) and redesigning the filter with the new impulse response leads to a new LP filter with cutoff frequency \( \omega_M = M\omega_0 \). One can say that tuning is achieved by “sampling” the taps. The resolution of such tuning is not very high and the filter characteristics deteriorate because \( \delta_0 \) and \( \delta_1 \) (Fig.1a) increase. The new deviations \( \delta_{M0} \) and \( \delta_{M1} \) are estimated in [18] as

\[ \delta_{M0} = M\delta_0; \quad \delta_{M1} = \delta_1 + (M - 1)\delta_0. \tag{32} \]

One possible way, proposed in [18], of improving the resolution is to realize the system as a multirate filter.

For the special case of a tunable linear-phase FIR notch filter, it is proposed in [19] to use quadratic programming to update the filter coefficients. The optimization procedure, is actually performed only once, during the initial design. Since a closed-form solution is achieved, the next tuning of the notch frequency and the notch depth does not require new optimizations, but only calculation of the new coefficients. These calculations, however, have a considerable volume, and include a matrix-vector multiplication. The method proposed in [20], in which only the notch width is tuned while the notch frequency is kept constant, has similar disadvantages.

4. Variable IIR Digital Filters

4.1. Additional problems and transformations

Usually, variable IIR digital filters are designed using the allpass transformations of Constantinides (1)-(6). Theoretically, it is possible to apply also the transformations of Oppenheim et al. (9), (13) on IIR prototypes in order to obtain variable IIR filters, but it is much more complex [13] and impractical. These transformations (in the forms (14), (16), (17)) have been used only by Estola [13] to obtain a specific variable structure containing three independent blocks – one all-zero, one all-pole and one overlap-add block. Although some of the coefficients of the transfer function are updated through simple calculations (within some limits), the method in [13] is complicated compared to other methods described hereafter.

Very interesting generalizations of both Oppenheim et al. [3] and Constantinides [2] transformations (both called “structural transformations” in [21]) are given in [22]. It is shown that while LP to LP transformation (3) is the best first-order transformation, there are problems in obtaining a variable BP filter with independent control of the passband edges \( \omega_0, \omega_2 \) (Fig.1b) and the bandwidth. Combining both transformations, Nie and Unbehauer [22] overcome these problems.

Most of the known publications, however, describe the use of the allpass transformations (3)-(5) only. It is possible to apply them in two ways: to replace all delay elements in the prototype LP filter by allpass sections of first-order (for LP and HP variable filters (3)) or of second-order (for BP and BS variable filters (4), (5)), or to transform the prototype transfer function and then synthesize the new (variable) circuit.

When the first approach (delay replacement) is applied for IIR prototypes, it very often produces delay-free loops. To avoid this, Johnson proposed to compute modified transfer function coefficients throughout the filtering process [9], but the volume of computations is so high, that the method is applicable only for variable filters that are rarely tuned. Steiglitz decreased the volume of computations to only about one multiplication, addition and division per coefficient, but
at the expense of, in fact, doubling the complexity of the filter [23]. To remove the delay-free-loop problem entirely, Schussler and Winkelkemper [1] suggested the use of FIR prototypes, but it was shown in [8] that the filter thus obtained is highly inefficient (with regard to the number of computations and the level of output noise), and a new method with modified coefficients, better than the one from [9], was proposed. The method is based on cascade or parallel connections of first- and second-order sections, but the volume of computations (including updating of the modified coefficients) is still very high. Many other possible ways to avoid the delay-free loops, such as removal through special transformations or the use of sample and hold devices [1] are discussed in [6], but they all appear to be impractical. The problem was resolved in [6] by using cascade or parallel realization of the transformed prototype with basic blocks of at most second-order (applying thus the approach suggested in [9]) and calculating their transformed coefficients in a separate device. In fact, Schussler and Kolb dealt with the poles and zeros of each basic block, instead of with the polynomial coefficients [6]. The main drawback, as in [9], is that the computation volume is quite high and special high-speed devices are required.

An efficient way of avoiding delay-free loops in the case of LP to BP and LP to BS transformations is to provide the factor $\frac{1}{\beta}$ in the numerator of $T(z)$ (4). This is achieved in [2] by taking $k=1$. Thus, a special case of LP to BP transformation (4) with guaranteed realizability is obtained:

$$T(z) = -z^{-1} \frac{z^{-1} - \frac{1}{\beta}}{1 - \frac{1}{\beta} z^{-1}}.$$  

However, having $k=1$ means (see (5)) that

$$\omega_2 - \omega_1 = BW = \omega_p,$$  

i.e., the bandwidth of the BP filter will be equal to the bandwidth of the prototype LP filter. As a result, $\omega_1$ and $\omega_2$ can no longer be controlled independently. This is the price to be paid for the elimination of the delay-free loops.

The restricted form of LP to BS transformation, eliminating all delay-free loops, appears as [2]

$$T(z) = z^{-1} \frac{z^{-1} - \frac{1}{\beta}}{1 - \frac{1}{\beta} z^{-1}} \text{ with } \omega_2 - \omega_1 = \frac{\omega_{sam}}{2} - \omega_p,$$

where $\omega_{sam}$ is the sampling frequency.

After applying (33) and (35) to a prototype LP filter of $N$-th order, i.e., adding $N$ more multipliers $\beta$, it becomes possible to tune the center frequency of the BP/BS filter by changing the single parameter $\beta$. The bandwidth, however, will stay constant and equal to $\omega_p$ (34).

One way of making the bandwidth tunable is to use a LP prototype with a single-parameter tunable cutoff frequency $\omega_p$ (34). This is possible only in a limited number of cases [24], [25] when the LP prototype is very simple (for example, of first-order or of higher order but containing equal first-order sections). The most general approach is to obtain a LP filter with a single-parameter tunable cutoff frequency by using LP to LP transformation (3) and then to apply LP to BP (33) or LP to BS (35) transformations on this LP filter. Thus, it becomes very important to find a way of applying (3) without getting delay-free loops. No such general technique has been known until now. Fortunately, an approximate method was advanced by Mitra et al. [26], and it has been successfully employed ever since [16], [27], [28]. This method is based on the Taylor series expansion of the LP to LP transformation (3) and can be explained as follows [28]. Let a second-order LP transfer function with zeros on the unit circle be

$$H(z) = \frac{1 + a_1 z^{-1} + z^{-2}}{1 + b_1 z^{-1} + b_2 z^{-2}}.$$

After applying LP to LP transformation (1), (3), we obtain

$$H'(z) = \gamma \frac{1 + a_1' z^{-1} + z^{-2}}{1 + b_1' z^{-1} + b_2' z^{-2}}.$$

where $\gamma$, $a_1'$, $b_1'$, and $b_2'$ all depend on $\beta$ and $a_1, b_1, b_2$, for example,

$$\gamma = \frac{1 - \beta a_1 + \beta^2}{1 - \beta b_1 + \beta^2 b_2}, \quad b_1' = \frac{b_1 - 2\beta(1 + b_2) + \beta^2 b_1}{1 - \beta b_1 + \beta^2 b_2}. \quad (38)$$

By expanding $\gamma$, $a_1'$, $b_1'$, and $b_2'$ in Taylor series with respect to $\beta$ and truncating after the linear term (assuming thus $\beta << 1$), we obtain very simple expressions such as

$$b_1' = b_1 + \beta c_1; \quad \gamma = 1 + \beta \rho,$$

where $c_1 = -2 - 2b_2 + b_1^2$ and $\rho = b_1 - a_1.$ \quad (39)

Fig. 3. Illustration of the tuning branches for $b_1'$ and $\gamma$ (39).

Then, we have two alternatives for tuning: either recalculate the coefficients $\gamma$, $a_1'$, $b_1'$, and $b_2'$ each time or introduce tuning branches with coefficients $\beta c_i$ in parallel with the nominal valued multipliers $a_1, b_1, b_2$, $\gamma = 1$ (assuming direct-form realization), as shown in Figs. 3a,b. Now, changing only $\beta$ in all of $\gamma$, $a_1'$, $b_1'$, and $b_2'$, we can tune the cutoff frequency (and the gain scaling factor $\gamma$) of the new section. Because this method is approximate, the tuning is linear and the shape of the magnitude response is not
changed only within a limited range of frequencies, usually several octaves.

Additional restricted forms are introduced in [2] by taking $\beta=0$ in (33) and (35). These forms yield arithmetically symmetric (with respect to the center frequency) BP and BS filters.

The second way of applying allpass transformations – first transform the prototype LP transfer function and then synthesize – appears to be impractical. It requires a huge amount of calculations (proportional to $N^2$, where $N$ is the order of the LP prototype) that cannot be performed in real-time applications.

4.2. Allpass transformations applied to classical (direct-form, lattice or ladder) IIR prototypes

Direct-form IIR prototypes are used in [1], [6], [9] for the application of nonconstrained transformations (3)-(5). The volume of computations necessary to recalculate the coefficients, is, however, so high ($O(N^2)$ operations for prototype of order $N$) that it is impossible to perform tuning in real time. The computation volume is decreased in [8] and [23], but the complexity of the filtering operation is almost doubled in [23]; in both cases, the operation “division” (which is difficult to realize) is included, and the resulting volume of computations is still high. It thus appears to be impossible to develop satisfactory efficient tunable filters using unconstrained transformations (3)-(5) with IIR prototypes and recalculation of the coefficients. The approach based on Taylor series expansion of the LP to LP transformation, advanced by Murakoshi et al. [28], is much more successful. They first factor the even-order LP transfer function into second-order products and then apply the procedure described by expressions (36)-(39) and Fig. 3 [28]. Besides the direct-form prototype, two other methods of design, based on lattice-form prototypes, are given in [28]. Only even-order transfer functions are discussed, because they cannot be realized with a real coefficient-parallel-allpass structure [26], which is considered to be better in the case of odd-order prototype transfer functions. The method in [28] is much better than all other methods using direct-form prototypes; it is possible to tune the cutoff frequency by changing a single parameter $\beta$ (38), (39)(Figs. 3a,b), and the tuning range is wider compared with complex coefficient variable filters based on the same Taylor series method [26]. It seems that the only disadvantage of the method is the change of the passband magnitude for some values of $\beta$.

An interesting approach to obtaining variable IIR filters is suggested by Erfani and Peikari [29]. It starts from a doubly terminated LC ladder structure; a concept of the generalized delay element [30] is used, and transformations similar to (3), (33) and (35), together with bilinear transformation, are introduced to finally obtain a variable digital filter without using any adaptors. This method, however, is not sufficiently straightforward, it is necessary to refer back to the analog domain for any change of the filter parameters, and $L$ and $C$ parameters from the LC-prototype participate in the formulae for coefficient updating. For these reasons, the method has not gained popularity. The method proposed in [31] is also quite similar, although the authors considered the filters to be wave digital.

4.3. Wave-digital-filter-based variable filters

The use of wave digital filters (WDFs) as prototype LP filters was first suggested in [5] and [6] as a possible way of avoiding delay-free loops after applying allpass transformations (3)-(5). A universal WDF structure was developed by El-Ghoroury and Gupta [5] for this purpose. The structure remains unchanged when the frequency properties are changed and, moreover, no reference back to the analog domain is necessary. Transformation (4) is also interpreted in a very interesting way in [5]: if we have one first- and one second-order allpass transformation,

$$T_i(z) = \frac{z^{-1} - (k-1)/(k+1)}{1 - z^{-1}(k-1)/(k+1)}; \quad T_2(z) = z^{-1} \frac{z^{-1} - \beta}{1 - \beta z^{-1}}.$$  (40)

then $T(z)$ (4) may be rewritten as

$$T(z) = T_i[T_2(z)].$$  (41)

The universal structure is obtained by applying this two-step transformation and thus avoiding the delay-free loops. Then, it becomes possible to tune the center frequency (by $k$) and the bandwidth (by $\beta$) independently. $k$ controls the values of $N+1$ adaptor multipliers, and these $N+1$ values must be recalculated together with the values of $N+2$ adaptor coefficients. $\beta$ appears the same in all of another group of $N$ adaptors. $k$ and $\beta$ themselves should be recalculated as given in (4) for every new $\omega_1$, $\omega_2$, and $\omega_0$. Thus, an entire arithmetic routine is required and the volume of calculations appears to be quite high, as in the other methods proposed in [6].

Another approach to realizing variable LP WDF without transformations was used in [32]. In this case, not only are all coefficients recalculated in the tuning process by referring to the analog prototype, but the computer also controls the filter structure. As a result, the volume of computations is even higher than in [5]. The HP WDF developed in [33] has the same drawbacks.

The most practical approach that permits tuning of LP WDF in real time has been developed by Watanabe et al. [34]. Even though the method is based on frequency scaling in the s-domain and the concept of the general delay element [30] is used, Watanabe et al. finally succeeded in implementing the cutoff frequency
scaling parameter \( \mu \) as a multiplier coefficient in the WDF [34]. This was achieved through Taylor series expansion of each adaptor coefficient \( \alpha_i \), represented as a function of \( \mu \) (i.e., \( \alpha_i(\mu) \)), and taking only the linear term:

\[
\alpha'_i(\mu) = \alpha_i(1) + (\mu - 1)\alpha'_i(1),
\]

where \( \alpha'_i(1) \) is the derivative of \( \alpha_i(\mu) \) evaluated at \( \mu = 1 \) and \( \alpha_i(1) \) is the value of the multiplier coefficient \( \alpha_i \) before scaling (changing) the cutoff frequency. Then (42) is realized in the same way as \( b' \) in Fig. 3. Hence, the cutoff frequency of the LP filter is tuned by changing a single value \( \mu \) in all multipliers \( \alpha_i \). As with all methods employing truncated Taylor series, tuning is successful in a limited frequency range, but because of the low sensitivity of the WDF, this range appears to be larger compared with other realizations [26], [28].

Second-order approximation of \( \alpha_i \) (truncation of the Taylor series after the second-order term),

\[
\alpha_i(\mu) = \alpha_i(1) + (\mu - 1)\alpha'_i(1) + 0.5(\mu - 1)^2\alpha''_i(1),
\]

was also investigated in [34]. The result is a much better frequency response, but the price is quite high – two more multipliers for each \( \alpha \). Therefore, (43) is recommended only for filters with a small number of coefficients \( \alpha \).

The excellent stability and sensitivity properties of the WDFs generally make them strong candidates for the development of variable and adaptive filters, but it seems, as mentioned in [35], that these properties are still not sufficiently used.

4.4. Parallel-allpass-structure-based variable filters

It is well known that all standard odd-order LP filters (polynomial and elliptic) can be realized as a parallel connection of two allpass structures (as shown in Fig. 4) by following the “allpass decomposition” [36]:

\[
G(z) = \frac{1}{2}[A_1(z) + A_2(z)]; H(z) = \frac{1}{2}[A_1(z) - A_2(z)].
\]

If \( G(z) \) is a LP filter, then \( H(z) \) is its doubly complementary HP filter. The structure has (under some easy conditions) very low passband sensitivity and good stability and, thus, like WDF, is also a strong candidate for a variable filter prototype. Direct applications of transformations (3)-(5), however, lead to delay-free loops, as in all other IIR prototypes. This was avoided in [26] and [27] by using truncated Taylor series expansion. For this, first \( A_1(z) \) and \( A_2(z) \) (Fig. 4) are decomposed into a cascade connection of first- and second-order allpass sections, and then LP to LP transform (3) is applied to each section. This is followed by Taylor series expansion and truncation, as illustrated with (36)-(39). Thus, a variable cutoff frequency LP filter with \( \alpha \) controlled by \( \beta \) is obtained. Note that the other filter, \( H(z) \) (Fig. 4), will also become variable. Then, BP/BS filters with variable center frequency are obtained by using (33), and these filters will have independent tuning of the bandwidth and the center frequency, each of them controlled by a single parameter. A signal processor implementation of such filters is reported in [16]. These types of variable filters are among the best known for odd-order LP prototype transfer functions (even-order LP prototypes produce complex coefficient realization), even though they are based on an approximate method.

If we take \( A_1(z)=1 \) in Fig. 4 and (44), then \( G(z) \) will be a BS (and \( H(z) - BP \)) transfer function, as shown in [24], [37] and [38]. If, additionally, \( A_2(z) \) is realized as a second-order lattice-based allpass section, very convenient second-order BP/BS filters are obtained with independent (orthogonal) tuning of the center frequency and the bandwidth. Such sections are used in [39], [40] to realize very efficient adaptive notch filters. This idea is further developed in [37] to obtain the structure shown in Fig. 5, having the transfer function [37], [38]:

\[
F(z) = \frac{1}{2}[1+A(z)] + \frac{1}{2}K[1-A(z)].
\]

Fig. 4. Parallel allpass structure realization.

If \( A(z) \) is a first-order allpass section, the circuit of Fig. 5 becomes a first-order (low-frequency) magnitude equalizer, and it is easily tuned by changing the allpass coefficient, similar to \( \beta \) in (3). If \( A(z) \) is a second-order allpass section, \( F(z) \) becomes a second-order magnitude equalizer of BP or BS type, depending on the value of \( K \) (Fig. 5). Implementing \( A(z) \) as a lattice structure ensures independent tuning of the center frequency, bandwidth and center-frequency gain. This structure is used in [41] to realize a dedicated-processor-based variable filter for audio applications.

4.5. Techniques based on multivariable polynomial representation of the filter coefficients

All methods of obtaining variable IIR filters using allpass transformations (3)-(5) appear to suffer from
some inherent limitations. As already shown, only the cutoff frequency of the LP/HP filters and the center frequency and the bandwidth of BP/BS filters can be tuned. It is impossible to tune the passband and stopband edges, (Fig. 1) or some other parameters of the magnitude response independently. Some more general methods that avoid such limitations have been advanced recently [42]-[44]. The main idea is to represent all \(2N+1\) filter transfer function coefficients \(a_i\) and \(b_j\) by polynomials of the type [42]

\[
b_j(\Psi) = \sum_{i_1=0}^{L_1} \sum_{i_2=0}^{L_2} \ldots \sum_{i_K=0}^{L_K} c_{j,i_1,i_2,\ldots,i_K} \prod_{m=1}^{K} (\Psi_m - \Psi_m^0)^{i_m}, \quad (46)
\]

where \(c_{j,i_1,i_2,\ldots,i_K}\) are coefficients of the polynomials \(b_j(\Psi); j=0, \ldots, 2N;\)
\(L_1, L_2, \ldots L_K\) are chosen polynomial orders;
\(K\) is the number of spectral parameters \(\Psi\) considered, such as cutoff frequency, bandpass and bandstop edges and transition bandwidth;
\(\Psi_m^0\) is the midinterval value of the \(m\)-th spectral parameter.

The main assumption is that if the frequency response changes only slightly, the corresponding changes of the filter coefficients will also be small. Thus, a smooth relationship between the coefficient values and the frequency response should exist. Therefore, if some set of fixed-parameter filters are optimally designed and their coefficients and spectral parameters are calculated, the coefficients necessary to realize spectral parameters and magnitude response between the two fixed-parameter filters can easily be calculated using (46). For this, first coefficients \(c_{j,i_1,i_2,\ldots,i_K}\) are computed using a curve fitting technique.

The final filter realization in [42] is of a cascade type. The method is valid and is really very general, but it requires a huge volume of calculations and can be implemented in real-time tuning only for filters of low order \(N\). An additional source of the higher volume of computations is the use of nonlinear minimization in the optimal design of the fixed filters and in the determination of the coefficients \(b_j(\Psi)\) (46). The computational burden is decreased to some extent in [43] and [44] by formulating the multivariable polynomial approximation problem as a linear problem. Moreover, in the method proposed in [43] and [44], the stability of the filter is controlled and guaranteed.

It should be pointed out that some ideas for linear interpolation of the variable filter coefficients were advanced in [45] much earlier than in [42]-[44].

Another technique for obtaining variable filters with arbitrary desired magnitude characteristics was developed by Deng and Soma [46]-[48]. It is based on sampling the given magnitude specification, constructing a multidimensional array with the samples, expanding this array as a sum of the outer products of vectors, and finally determining the filter coefficients by using one-dimensional instead of multivariable (46) polynomials. Then a special parallel structure, containing cascaded changeable and fixed parts in each parallel branch, is used to realize the desired magnitude characteristic. This method, like the others described in the present section, is too complicated and computationally intensive to be practical.

### 4.6. High-accuracy tunable IIR filters

High resolution of the tunable parameters is required in many applications. The interrelation between the tuning accuracy and the sensitivity of the transfer function coefficients was considered in [33] and [34], and it was pointed out that lower sensitivity will ensure higher accuracy. It is clear, also, that lower sensitivity will permit shorter multiplier coefficient word-length for a given accuracy, i.e., faster computations and updating of the tunable filter coefficients. In the case of variable filters based on WDF and truncated Taylor series expansion, the lower sensitivity of the WDF widens effectively the frequency range for which the first-order approximation (41) holds and the characteristics do not degrade [34].

It is clear, so far, that the following strategy should be used in order to obtain a high-tuning-accuracy variable filter when employing allpass transformations [25]:

- select or develop a LP prototype with low sensitivity in order to ensure high accuracy of the bandwidth tuning of the filter;
- select or develop allpass sections with low sensitivity in the frequency range within which the center frequency of the new BP/BS filter will be tuned. Using these allpass sections for transformations (33) and (35) (and in some cases (4) and (5)) will provide high accuracy in the tuning of the center frequency.

Following this strategy, very low-sensitivity tunable second- and fourth-order BP/BS filter sections (performing much better than those from [24]) were developed by Stoyanov and Kawamata [25]. Moreover, the BP and BS transfer functions are obtained at different outputs of a single universal section, as shown in Fig. 6 for the second-order section. Low-sensitivity allpass and LP sections can be selected in [49]-[51] or might be developed by using some sensitivity minimization techniques, like those in [52] and [53]. For the section shown in Fig. 6 (which is optimized for low-frequency applications) it is additionally possible to provide high tuning accuracy in the high-frequency range (near half the sampling frequency) merely by changing the sign at some of the adders, as indicated by signs in brackets, and one multiplier value from \(b\) to \(c\).
4.7. Complex coefficient variable filters

Complex coefficient digital filters have been gaining popularity in recent years due to their advantages in processing both real and complex signals (often encountered in telecommunications) and in providing very low passband sensitivity. In fact, if \( G(z) \) is an even-order LP transfer function, the decomposition (44) will produce \( A_1(z) \) and \( A_2(z) \) with complex coefficients. It is well known [54] that in this case, it is sufficient to realize only \( A_1(z) \) in Fig. 4 and the complementary LP/HP or BP/BS filters will appear, respectively, at the real and the imaginary output of the realization. Such a structure was turned into a variable in [27] by applying LP to LP transformation (3) and then using truncated Taylor series, as explained by (36)-(39). The results are similar to those for odd-order LP transfer functions (Section 4.4) but the cutoff frequency linear tuning range is narrower. In this sense, the circuits developed in [34] for even-order LP filters are superior.

Another approach to developing variable complex IIR filters was proposed by Murakoshi and Nishihara [55]. It is based on a circuit transformation proposed in [56], which is able to turn any real circuit into a complex one. By applying this together with LP to LP transformation (3) on a real coefficient LP filter, a complex LP filter with variable cutoff frequency is obtained. If instead of (3), some new transformations, derived in [55], are used, variable complex BP filters with tunable bandwidth and, at the same time, one of the passband edge frequencies remaining fixed, will be obtained. Variable BS filters with the same properties can also be obtained in this manner. Obviously, this method is much more universal than the one in [27] and most of those already discussed. While the method in [27] is approximate, i.e., tuning without degradation of the characteristics is possible within some limited frequency range, the method in [55] does not have such limitations. The complexity of the LP filter is the same as that of all complex coefficient filters. The variable BP filter in [55] employs, however, more elements, since the new transformation requires one rotation circuit and one first-order complex coefficient allpass section per each pair of delay elements.

4.8. Other types of variable IIR filters

In this section we shall discuss some variable filters that are not obtained by any of the transformations already described. Usually, these are simple second-order BP or BS filter sections, used as basic blocks in adaptive filters for tracking of a single or multiple sinusoids. The most popular among them are the sections proposed in [56] and further investigated in [57], [58]. They are obtained from active RC state-variable biquads and are not canonical with respect to the multiplier number (3 multipliers for second-order sections). Under certain conditions it becomes possible to control the center frequency and the bandwidth of the BP sections almost independently. They have low coefficient sensitivity when designed for low-frequency applications, thus providing higher tuning resolution in the low-frequency range. These sections are widely used (see, for example, [59] and [60]) with fixed bandwidth and tunable center frequency to realize notch filters for adaptive detection and enhancement of multiple sinusoids. It should be mentioned, however, that the section shown in Fig. 6 might be much better for such applications. It has unconditional (not approximate like the circuit in [59]) orthogonal tuning of the center frequency and the bandwidth, low sensitivity not only for low, but also for high frequencies, and provides a BS output, whereas in [56]-[59], an additional adder is employed in order to realize the same output.

Other examples of this type are the BP/BS second-order sections proposed in [61], [62]. They are, however, based on direct-form realization and thus employ too many multipliers.

An interesting modification of the standard second-order BS transfer function is proposed in [63] in order to introduce independent control of the passband gain.

Many other variable sections of this type can be found in the literature on adaptive filters, but they are usually interesting only as single sections and cannot be used for general design of variable filters.

5. Two-Dimensional Variable Digital Filters

Most of the theory of the 1-D variable filters can be carried over to the 2-D case. There are, however, many specific difficulties and additional problems connected with the implementation. Consequently (and also because the need for such filters was not so strong), the theory and design of variable 2-D filters is not very well developed and the number of publications is not so large. Moreover, those publications are usually a kind of extension of some 1-D technique (by the same authors).
Spectral transformations, similar to those of Constantinides (3)-(5), were first introduced in [64] and it was shown that \( z_1 \) and \( z_2 \) should be replaced by 2-D allpass transfer functions. Note, however, that many different transformations are possible, e.g., two-variable transformation of the first- or higher order,

\[
z(t) = T(z_1, z_2) = \frac{(\gamma_t + \beta z_1 + \alpha z_2 + z_1 z_2)}{1 + \alpha z_1 + \beta z_2 + \gamma_t z_1 z_2},
\]

and single-variable transformation of the first- or higher order,

\[
z_1 = \frac{a_1 + z_1}{1 + a_1 z_1}, \quad z_2 = \frac{a_2 + z_2}{1 + a_2 z_2}.
\]

There are 10 transformations of only first and second order given in [64], including combinations of first and second orders. It is clear that even first-order two-variable transformations (47) will considerably increase the complexity of the transformed circuit; that is why it is recommended to use only single-variable first-order transformations such as (48) for the design and realization of variable 2-D filters. Even in that case, some new problems, such as a requirement for additional memory, arise and, as seen from (48), the number of multipliers also increases. There are some other ideas in [64] about how to realize tunable 2-D filters using spectral transformations. An alternate way of obtaining these transformations is given by Erfani et al. [65], where, as in [29] and [30], they start from analog reactances.

The transformation of Oppenheim et al. was also extended to the 2-D case in [66] and [67], but was combination with McClellan transformation [68] by converting the 1-D FIR filter to 2-D:

\[
\cos \omega = P \cos \omega_1 + Q \cos \omega_2 + R \cos \omega_1 \cos \omega_2 + S.
\]

A variable LP FIR 2-D filter was obtained in [66] by starting from a linear-phase 1-D prototype (8) and applying first (49) with \( P=Q=R=0.5 \) and then the first-order Oppenheim et al.'s transformation (10). The result is [66]

\[
H(e^{i\Omega_1}, e^{i\Omega_2}) = 0.5 \sum_{n=0}^{N} a(n)[1 + (A_0 + A\cos \Omega_1) + (B_0 + B\cos \Omega_2)(B_0 + B\cos \Omega_2)]
\]

where \( A_0 \) and \( B_0 \) are the analogs of \( A_0 \) and \( A_1 \) in (10) for the second frequency axis. Under the condition \( A_0 + A_1 = B_0 + B_1 = 1 \), the frequency characteristics along the axes \( \Omega_1 \) and \( \Omega_2 \) are controlled (and tuned) independently by \( A_0 \) and \( B_0 \), respectively.

A similar approach is used in [67] to obtain a variable 2-D BP filter. The difference is that (10) is first applied to obtain a 1-D variable BP filter and then is converted to 2-D using (49) and some results from [11]. There are some problems in tuning the center frequency while keeping the bandwidth constant, which are overcome by using third-order transformation (9). The price of this is a considerable increase in the number of delay elements used.

An interesting approach to designing variable cutoff boundary 2-D IIR filters was developed in [69]. The 2-D extension of the Oppenheim et al. transformation (as utilized in (50)) was used to convert the numerator of the given 2-D IIR LP prototype, and single-variable first-order spectral transformation (48) was employed for the denominator. The method works, but the tuning is not exact and only LP filters can actually be directly designed.

In [70] and [71] another method of designing 2-D LP IIR filters was proposed. It starts from an analog prototype with a product separable denominator, and the digital transfer function is obtained by application of the bilinear transform. The method is rather unclear and the authors discuss mainly the stability problems, and thus it is difficult to compare this method with others.

The method of designing 1-D variable FIR filters using 2-D filters, given in [17] and described in Section 3, is also extended to the 2-D case in [72], [73]. The principle is the same (obtaining an M-D filter from an (M+1)-D one), and thus no additional explanations are necessary. This method is better than most of the others since it is more precise and permits full control of the band-edge frequencies and of the maximum passband and stopband deviations. There are two main drawbacks: a huge amount of initial calculations to design the 3-D prototype and a larger amount of calculations for coefficient updating are necessary, compared to methods using the McClellan transformation (49). Therefore, similarly to the 1-D case, this method is recommended for filters that are only occasionally tuned.

The methods in Section 4.5, based on different multivariable polynomial representations of the filter coefficients [42]-[48], have been extended to 2-D cases in [74]-[78]. The method in [74] has the same merits and disadvantages as the one in [42], as discussed in Section 4.5. This method was also investigated in [73]. The method from [48] was further developed in [75] for 2-D variable IIR filters. It uses a set of 1-D constant filters and requires approximation of several 1-D polynomials. It is computationally efficient, but is constrained to only quadrantly symmetric magnitude characteristics. In [76] and [77] more powerful decomposition, using 2-D constant filters and 1-D polynomial approximations, is proposed. In [78] additional “real-complex” decomposition is introduced, which ensures not only variable magnitude characteristics, but also a linear phase. This method also uses 2-D constant filters and 1-D polynomial approximation. All methods in [75]-[78] result in complicated parallel structures (corresponding to the decompositions) and
still require many calculations, making the methods infeasible for real-time tuning applications.

6. Implementations

Many practical implementations of variable digital filters have been reported in the literature. Very often, standard digital signal processors (DSP) are used and different aspects of their applications are discussed. Such realizations of parallel-allpass-structure-based variable filters, using TMS 320C10, are described in [16] and [27]. Bandpass sections, obtained through constrained allpass transformations (33), (35), are experimentally investigated in [24] using TMS 320C25. Additional information about parallel allpass and WDF implementations using DSP can be found in [79]. More often, however, variable digital filters are realized as dedicated chips for specific applications. One of the first and most famous of such implementations was that of the WDF-based set of chips for telecommunication applications (subscriber-line interface circuits), described in [35] and [80]. A dedicated processor for audio applications (based on the structure from Fig. 5) is reported in [41], and for video signals, in [81]. The filter in [81] is very fast (40 MSamples/s) and can also be used for realization of 2-D separable filters, but tuning (coefficient reprogramming) cannot be done during filtering. A very powerful tunable system of filters and equalizers for digital audio is described in [82]. The chip contains only low-sensitivity structures, such as WDF and sections with minimized sensitivity (structures of Agarval and Burrus) and can realize 49\textsuperscript{th} x 3 different filter and equalizer characteristics. It uses canonical signed digit (CSD) number representation, which is actually 3-positional. The use of CSD is criticized in [83] as inappropriate for real-time tuning applications because additional time is required for coding of the coefficients. In [83], the use of bit-level coding of the input signals, together with a modified Booth algorithm for the multiplications, is proposed in order to reduce the computation costs. The FIR filter so obtained permits tuning in real time. It should be mentioned, however, that CSD is employed in many other implementations (see, for example, [84]).

In contrast to the variable system in [82], many other filters for audio applications, proclaimed to be variable, actually have only several fixed characteristics (for example, 5 magnitude equalizers for 5 different types of music or 4 different LP filters for different types of surroundings [85], one of which can be selected).

All contemporary trends in the design and production of digital systems may be found in the recent publications on variable and programmable filters. Special attention is paid to the new more efficient architectures using pipelining and interleaving [86], [87], and to the new CMOS-based VLSI technologies [84], [86]. The extremely high speeds of 85 MHz for IIR [86] and 100 MHz for FIR [88] filter realizations have already been achieved.

Many variable digital filters have been patented. Some of them have specific structures, like the one in [89] that is tuned by selecting some of many possible branches in a parallel allpass structure, and others [90] based on something like the direct-form LP section. There are state-space-section-based variable filters [91] and even variable sampling rate filters [92], [93]. Note that tuning the cutoff frequency by changing the sampling rate is one of the oldest ideas for variable digital filter realizations.

7. Concluding Remarks

An attempt to review the theory, all known techniques, and the present situation in the field of the variable digital filters, was made in this paper. It has gradually become apparent that this field is too vast to be easily overviewed and described in a single article. Consequently, some publications were not included, some structures and techniques have been mentioned only very briefly and some comparisons were quite superficial. No details were given on the new architectures and technologies or on many interesting implementations and applications. Most of the publications listed in the references are, however, a good starting point for additional deeper study. Moreover, it was difficult to discuss variable digital filters without entering another vast field, that of adaptive filters.

It seems that the field of 1-D variable filters is quite mature, therefore no new significant ideas can be expected. There is some room for additional improvement of the resolution and for guaranteeing stability during tuning. With the decreasing cost of computations and increasing speed, it may be anticipated that some of the most complicated methods, like those using multivariable polynomial representation of the coefficients, may gradually become practical, and new methods of this type might be developed. A massive penetration of variable digital filters and equalizers in home audio and video systems will be witnessed soon.

Concerning 2-D variable filters, there is much to be done and many new methods, structures and transformations will probably be proposed. The main motivation and driving force for research in this direction will be the clear need for variable systems in many multimedia applications.

References


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