A NEW STATISTICAL SENSITIVITY OF LINEAR DISCRETE-TIME STATE-SPACE SYSTEMS

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ABSTRACT
This paper proposes a new statistical sensitivity of linear discrete-time state-space systems. The proposed sensitivity of state-space systems agrees with the actual output error variance since it is derived from the exact evaluation of the output error variance due to coefficient deviation. The sensitivity in this paper is represented by the controllability and the observability gramians and the state covariance matrix of the system. When the variance of coefficient variations is very small, the proposed sensitivity is identical to the conventional statistical sensitivity of state-space systems. This paper also proposes a method of synthesizing minimum sensitivity structures. Numerical examples show that the proposed sensitivity is in very good agreement with the actual output error variance, and that minimum new statistical sensitivity structures have a very small degradation of the frequency characteristic due to coefficient quantization.

1. INTRODUCTION
Digital signal processing is used in such areas as data communication, digital audio, speech processing, biomedical data processing, and a host of other applications. One of the most valuable aspects of digital signal processing is digital filters and digital controllers. They are implemented with finite wordlength binary digits on general purpose computers or with special-purpose hardware. The actual system transfer function with finite wordlength always deviates from the ideal system transfer function due to the coefficient quantization.

The problems of quantization effects have been studied in detail by the state-space approach in [1–7]. The state-space approach has been attempted for linear discrete-time systems from the view point of frequency domain [1, 6–9] and time domain [3, 5, 10].

This paper is devoted to improve the statistical sensitivity measure defined in [3, 5, 10] by the authors. It is based on the assumption that the coefficients are stochastically varied. This conventional statistical sensitivity is represented by the controllability gramian and the observability gramian. It is well known that the class of minimum conventional sensitivity structures includes balanced realizations as a special case. However, this conventional statistical sensitivity is derived with the approximation in order to simplify the analysis of the output error variance. Therefore, it is desirable to derive a sensitivity measure without this approximation.

This paper is organized as follows: Section 2 introduces the definition of the statistical sensitivity. Section 3 proposes a new statistical sensitivity of linear discrete-time state-space systems. The proposed sensitivity is derived from the exact evaluation of the output error variance due to coefficient deviation without approximation. It is represented by the controllability and the observability gramians and the state covariance matrix of the system. If the variance of coefficient variations is very small, the proposed sensitivity is identical to the conventional statistical sensitivity of state-space systems. Section 4 considers a method of synthesizing minimum sensitivity structures. Section 5 gives two numerical examples to demonstrate that the proposed sensitivity is in very good agreement with the actual output error variance, and that minimum new statistical sensitivity structures have a very small degradation of the frequency characteristic due to coefficient quantization.

2. DEFINITION OF STATISTICAL SENSITIVITY
For a given n th order transfer function \( H(z) \), a linear discrete-time system can be described by the following state equations:

\[
\begin{align*}
x(k + 1) &= Ax(k) + bu(k) \\
y(k) &= cAx(k) + du(k)
\end{align*}
\]

where \( u(k) \) is the scalar input, \( y(k) \) is the scalar output, and \( x(k) \) is the n th order state vector; matrices \( A \), \( b \), \( c \) and \( d \) are \( n \times n \), \( n \times 1 \), \( 1 \times n \), \( 1 \times 1 \) real constant matrices. The
signal flowgraph representation of a 2nd order state-space system is shown in Fig. 1. The transfer function $H(z)$ is described in terms of coefficient matrices as

\[ H(z) = c(zI - A)^{-1}b + d. \]  

(3)

The transfer function is invariant under nonsingular transformation matrices $T$ of the state, i.e., if $x' = T^{-1}x$, then the new realization is $(T^{-1}AT, T^{-1}b, cT, d)$.

The actual systems can be implemented only to a finite degree of accuracy because of physical constraints, for example, the finite wordlength implementation in digital filters [1–4, 10], and the linear approximation in variable digital filters [11]. In [3, 10], the authors proposed the statistical approach to analyze the coefficient variations in digital filters. In the following analysis, we adopt the statistical approach to analyze the coefficient variations in digital filters [11]. In [3, 10], the authors proposed the statistical approach to analyze the coefficient variations in digital filters. In the following analysis, we adopt the statistical approach [3, 5, 10], since the statistical approach can simplify the analysis of the error variance and derive useful theoretical results.

To get a statistical sensitivity, we assume that an actual system corresponding to the ideal system $(A, b, c, d)$ is described by the following state equations:

\[ \ddot{x}(k + 1) = [A + \Delta A(k)]\ddot{x}(k) + [b + \Delta b(k)]u(k) \]  

(4)

\[ \ddot{y}(k) = [c + \Delta c(k)]\ddot{x}(k) + [d + \Delta d(k)]u(k) \]  

(5)

where $\Delta A(k), \Delta b(k), \Delta c(k),$ and $\Delta d(k)$ are error matrices, the variation $\Delta a_{ij}(k)$ in the error matrix $\Delta A(k)$ is a white noise with zero mean and variance $\sigma^2$, and the input $u(k)$ is a white Gaussian noise with zero mean and unit variance. The assumptions of the variations $\Delta b_i(k), \Delta c_i(k),$ and $\Delta d_i(k)$ are the same as that of $\Delta a_{ij}(k)$ [10].

The statistical sensitivity $S$ is defined as the output error variance $E[\Delta y^2]$ normalized by the variance $\sigma^2$ of coefficient variations [5, 10]:

\[ S = \frac{E[\Delta y^2]}{\sigma^2} = \lim_{k \to \infty} \frac{E[(\ddot{y}(k) - y(k))^2]}{\sigma^2} \]  

(6)

In [5, 10], to simplify the analysis of the output error variance, the statistical sensitivity is derived with the approximation based on the assumption that the variance $\sigma^2$ is very small. This approximated statistical sensitivity is called conventional statistical sensitivity $S_c$ in this paper. It can be represented as follows [5, 10]:

\[ S_c = \{\text{tr}[K] + 1\}\{\text{tr}[W] + 1\} \]  

(7)

where the controllability gramian $K$ and the observability gramian $W$ are the solutions of the following Lyapunov equations, respectively:

\[ K = AKA^T + bb^T \]  

(8)

\[ W = A'WA + c'e. \]  

(9)

The conventional statistical sensitivity $S_c$ can be transformed into $S'_c$ by a nonsingular matrix $T_0$ as

\[ S'_c = S_c(T_0^{-1}AT_0, T_0^{-1}b, cT_0, d) \]  

\[ = \{\text{tr}[T_0^{-1}KT_0^{-1}] + 1\}\{\text{tr}[T_0^{-1}WT_0] + 1\} \]  

(10)

It is well known that the minimum conventional sensitivity structures satisfy $K' = W'$. Therefore, a class of minimum conventional sensitivity structures includes balanced realizations as a special case [10].

3. A NEW STATISTICAL SENSITIVITY OF LINEAR DISCRETE-TIME STATE-SPACE SYSTEMS

3.1. Derivation of the output error variance

To get the output error variance $E[\Delta y^2]$, we first analyze the output error due to coefficient variations.

Subtracting (2) from (5), and subtracting (1) from (4), the output error $\Delta y(k) = \ddot{y}(k) - y(k)$ is obtained as follows:

\[ \Delta y(k) = c\ddot{v}(k) + \Delta c(k)x(k) \]  

+ $\Delta e(k)v(k) + \Delta d(k)u(k)$ \]  

(11)

\[ v(k + 1) = A\ddot{v}(k) + \Delta A(k)x(k) \]  

+ $\Delta A(k)v(k) + \Delta b(k)u(k)$ \]  

(12)

where $\ddot{v}(k) = \ddot{x}(k) - x(k)$ is the state error vector, and the higher order terms $\Delta A(k)v(k)$ and $\Delta c(k)v(k)$ are not ignored in order to evaluate the output error variance exactly. Fig. 2 gives the signal flowgraph representation of the output error $\Delta y(k)$. 

![Figure 1: Signal flowgraph representation of 2nd order state-space equations.](image-url)
From (11), we obtain the output error variance \( E[\Delta y(k)^2] \) as
\[
E[\Delta y(k)^2] = E[\{cV(k) + \Delta c(x(k) + \Delta cV(k) + \Delta d(k)u(k)\}\cdot \{cV(k) + \Delta c(x(k) + \Delta cV(k) + \Delta d(k)u(k)\}^T] \\
= E[c^2V(k)c^T(k)] + E[\Delta c(x(k)x^T(k)\Delta c^T(k)] \\
+ E[\Delta c(x(k)V(k)u^T(k)\Delta c^T(k)] + E[\Delta d^2(k)u^2(k)], \]
(13)

Letting
\[
X(k) = E[x(k)x^T(k)] \\
V(k) = E[v(k)v^T(k)]
\]
(14, 15)
yields
\[
E[\Delta y(k)^2] = cVe^c + \sigma^2 tr[X] + \sigma^2 tr[V] + \sigma^2. \]
(16)
Thus, we have the output error variance \( E[\Delta y^2] \) in the steady state as
\[
E[\Delta y^2] = cVe^c + \sigma^2 tr[X] + \sigma^2 tr[V] + \sigma^2. \]
(17)
where \( X \) and \( V \) are the solutions of the following Lyapunov equations (see Appendix):
\[
X = AXA^T + bb^T \\
V = AVA^T + \sigma^2 \{tr[X] + tr[V] + 1\} I. \]
(18, 19)
Using (17), the new statistical sensitivity can be obtained as
\[
S = E[\Delta y^2]/\sigma^2.
\]

### 3.2. Relation between the new statistical sensitivity and the gramians

We now consider the representation of (17) in terms of the controllability gramian \( K \) and the observability gramian \( W \). The first term \( eVe^c \) in (17) can be rewritten as
\[
eVe^c \\
= tr[ecVe^c] \\
= tr \left[ c \sum_{j=0}^{\infty} A^j \sigma^2 \{tr[K] + tr[V] + 1\} (A^T)^j c \right] \\
= \sigma^2 \{tr[K] + tr[V] + 1\} tr \left[ \sum_{j=0}^{\infty} (A^T)^j c^T A^j \right] \\
= \sigma^2 \{tr[K] + tr[V] + 1\} tr[W]. \]
(20)
From (8) and (18), the matrix \( X \) equals the controllability gramian \( K \). From (19), the matrix \( V \) can be represented as follows:
\[
V = AVA^T + \sigma^2 \{tr[X] + tr[V] + 1\} I \\
= \sum_{j=0}^{\infty} A^j \sigma^2 \{tr[K] + tr[V] + 1\} (A^T)^j \\
= \sigma^2 \{tr[K] + tr[V] + 1\} \sum_{j=0}^{\infty} A^j (A^T)^j \\
= \sigma^2 \{tr[K] + tr[V] + 1\} R \]
(21)
where the matrix \( R \) is the solution of the following Lyapunov equation:
\[
R = ARA^T + I. \]
(22)
From (21), the third term \( tr[V] \) in (17) can be rewritten as
\[
tr[V] = \sigma^2 \{tr[K] + tr[V] + 1\} tr[R] \\
= \sigma^2 \{tr[K] + 1\} \frac{tr[R]}{1 - \sigma^2 tr[R]}, \]
(23)
Substituting (20) and (23) into (17), we obtain the expression of the output error variance \( E[\Delta y^2] \) as
\[
E[\Delta y^2] = \sigma^2 \{tr[K] + 1\} \{tr[W] + 1\} \left\{ 1 + \frac{\sigma^2 tr[R]}{1 - \sigma^2 tr[R]} \right\} \\
= \sigma^2 \{tr[K] + 1\} \{tr[W] + 1\} \left\{ 1 - \frac{1}{1 - \sigma^2 tr[R]} \right\}. \]
(24)
Thus, the expression of the new statistical sensitivity \( S \) is obtained as
\[
S = E[\Delta y^2]/\sigma^2 \\
= \{tr[K] + 1\} \{tr[W] + 1\} \left\{ 1 - \frac{1}{1 - \sigma^2 tr[R]} \right\}. \]
(25)

The above sensitivity measure implies that the proposed sensitivity can be applied in the state-space systems of which variance \( \sigma^2 \) of coefficient deviations is large, for example, the variable digital filters implemented with linear approximation [11].

Note that if \( \sigma^2 tr[R] \ll 1 \), the proposed sensitivity can be approximated as
\[
S \approx \{tr[K] + 1\} \{tr[W] + 1\} \{\sigma^2 tr[R] + 1\}. \]
(26)
Furthermore, if \( \sigma^2 \to 0 \), from which we have \( \sigma^2 tr[R] \to 0 \), the proposed sensitivity is identical to the conventional statistical sensitivity \( S_c \) as
\[
S = S_c = \{tr[K] + 1\} \{tr[W] + 1\}. \]
(27)
4. SYNTHESIS OF MINIMUM SENSITIVITY STRUCTURES

We now consider the minimization problem of the statistical sensitivity. The statistical sensitivity $S$ can be transformed into $S'$ by a nonsingular matrix $T$ as follows:

$$S' = S(T^{-1}A', b, c', d) = S(T^{-1}AT, T^{-1}b, cT, d)$$

where

$$\{\tr[T^{-1}KT^{-1}]+1\}{\{\tr[T^iWT]+1\}\{1-\sigma^2\tr[R(P)P^{-1}]\}^{-1}}\cdot\{1-\sigma^2\tr[R(P)P^{-1}]\}^{-1}$$

(28)

where $P = TT'$ and $R(P)$ is the solutions of the following Lyapunov equation:

$$R(P) = AR(P)A^T + P.$$  (29)

The statistical sensitivity minimization problem is to find an equivalence transformation matrix $T$ which minimizes $S'$. Therefore, we consider an equilibrium point of the following matrix differential equation

$$\frac{\partial S'}{\partial P} = -\{T^{-1}KP^{-1}\}{\{\tr[WW^T]+1\}\{1-\sigma^2\tr[R(P)P^{-1}]\}^{-1}}\cdot\{1-\sigma^2\tr[R(P)P^{-1}]\}^{-1}$$

(30)

where $M(P)$ is the solution of the following Lyapunov equation:

$$M(P) = A'M(P)A + P^{-1}.$$  (31)

Letting $\frac{\partial S'}{\partial P} = 0$ yields

$$P'G(P)P = H(P)$$  (32)

where

$$G(P) = \{\tr[KP^{-1}]+1\}{\{1-\sigma^2\tr[R(P)P^{-1}]\}^{-1}}\cdot\{\tr[WW^T]+1\}{\{1-\sigma^2\tr[R(P)P^{-1}]\}^{-1}}$$

$$H(P) = \{\tr[KP^{-1}]+1\}{\{1-\sigma^2\tr[R(P)P^{-1}]\}^{-1}}\cdot\{\tr[KP^{-1}]+1\}{\{1-\sigma^2\tr[R(P)P^{-1}]\}^{-1}}.$$  (33)

We cannot derive an explicit expression of the solution $P$ from (32), and show that $S'$ has a unique global minimum. However, we can obtain the solution $P$ by the following iterative procedure using gradient algorithm [12]:

$$P_{i+1} = P_i - \mu \frac{\partial S'}{\partial P} |_{P=P_i} \quad (34)$$

Figure 2: Signal flowgraph representation of the output error $\Delta y(k)$.

### Table 1: Coefficient Matrices for Example 1 ($d = 0.0495$).

<table>
<thead>
<tr>
<th>Structure</th>
<th>$A$</th>
<th>$b$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Direct</td>
<td>0.0</td>
<td>1.0</td>
<td>0.0</td>
</tr>
<tr>
<td>form II</td>
<td>0.0</td>
<td>0.0</td>
<td>1.0</td>
</tr>
<tr>
<td></td>
<td>0.1377</td>
<td>-0.6959</td>
<td>1.1619</td>
</tr>
</tbody>
</table>

where $P_i = T_iA_i$ is the solution of the $i$ th iteration, and $\mu$ is a positive step size. The initial solution $P_0 = T_0A_0$ is determined by a nonsingular transformation matrix $T_0$ such as $T_0^{-1}KT_0^{-1} = T_0^iWT_0$; this means that the initial realization $(T_0^{-1}AT_0, T_0^{-1}b, cT_0, d)$ is a balanced realization.

5. NUMERICAL EXAMPLES

**Example 1:**

To illustrate the validity of the analysis of the output error variance, we now consider the following transfer function:

$$H(z) = 0.04953 + 0.1485z^{-1} + 0.1485z^{-2} + 0.04953z^{-3}$$

$$1 - 1.1619z^{-1} + 0.6959z^{-2} - 0.1377z^{-3}.$$  (35)

The coefficient matrices of the direct form are shown in Table 1. Fig. 3 gives the proposed statistical sensitivity, the conventional statistical sensitivity, and the results of simulation for the normalized output error variance $E[\Delta y^2]/{\sigma^2}$ due to coefficient variations, where the time average is used in stead of the statistical average, and $\infty$ of (6) is approximated to $3 \times 10^5$. It is seen from Fig. 3 that the proposed

Table 2: Coefficient Matrices for Example 2.

<table>
<thead>
<tr>
<th>Structure</th>
<th>A</th>
<th>b</th>
<th>c^l</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parallel form</td>
<td>0.0 1.0 0.0 0.0</td>
<td>0.0 1.0 0.0 0.0</td>
<td>-0.006105 0.012723</td>
<td>0.030753</td>
</tr>
<tr>
<td></td>
<td>-0.90845 1.9019 0.0 0.0</td>
<td>1.0 0.01031</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.0 0.0 0.0 1.0</td>
<td>0.0 0.01031</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.0 0.0 -0.9794 1.9634</td>
<td>1.0 -0.01263</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Balanced form</td>
<td>0.9876 0.0176 0.0821 -0.0125</td>
<td>0.12976 0.12976</td>
<td>0.030753</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-0.0176 0.9440 -0.0219 -0.0847</td>
<td>0.10386 -0.10386</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>-0.0821 -0.0219 0.9728 0.0772</td>
<td>0.15860 -0.15860</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>-0.0125 0.0847 -0.0772 0.9613</td>
<td>0.13854 0.13854</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Minimum sensitivity form ((\sigma^2 = 0.5 \times 10^{-4}))</td>
<td>0.9875 0.08215 0.01248 -0.01757</td>
<td>-0.1298 -0.1298</td>
<td>0.030753</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-0.08215 0.9727 -0.07717 0.02190</td>
<td>-0.1588 0.1588</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.01248 0.07717 0.9614 0.08462</td>
<td>0.1387 0.1388</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.01757 0.02190 -0.08462 0.9440</td>
<td>0.1038 -0.1038</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3: Statistical Sensitivity for Example 2 (\(\sigma^2 = 0.5 \times 10^{-4}\)).

<table>
<thead>
<tr>
<th>Structure</th>
<th>(\sigma )</th>
<th>(\sigma^c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parallel form</td>
<td>9.694 \times 10^3</td>
<td>5.213 \times 10^4</td>
</tr>
<tr>
<td>Balanced form</td>
<td>8.485095</td>
<td>8.430484</td>
</tr>
<tr>
<td>Minimum sensitivity form ((\mu = 0.3))</td>
<td>8.485071</td>
<td>8.430584</td>
</tr>
</tbody>
</table>

Sensitivity agrees with the output error variance even if the variance of coefficient deviations is large, and that the conventional statistical sensitivity agrees with the output error variance when the variance of coefficient deviations is toward zero.

Example 2:

To illustrate the effectiveness of the synthesis method proposed, we now consider the following transfer function:

\[
H(z) = \frac{0.03075 - 0.1187z^{-1} + 0.1761z^{-2} - 0.1187z^{-3} + 0.03075z^{-4}}{1 - 3.8656z^{-1} + 5.6229z^{-2} - 3.6468z^{-3} + 0.8897z^{-4}}
\] (36)

Table 2 shows the coefficient matrices of the parallel, balanced, and minimum sensitivity forms. Then, \(\mu = 0.3\) and the iteration number is 2. Table 3 shows the new statistical sensitivity \(\sigma^c\) of these structures, and the conventional statistical sensitivity \(\sigma\) of these structures.

The frequency responses of these of the 8 bits truncated versions are shown in Fig. 4. It is seen that minimum sensitivity structure has a very small degradation due to coefficient quantization.

6. CONCLUSIONS

This paper has proposed the new statistical sensitivity of linear discrete-time state-space systems. The proposed sensitivity is derived from the exact evaluation of the output error variance due to coefficient deviation. It is represented by the controllability and the observability gramians and the state covariance matrix of the system. This paper has also proposed a method of synthesizing minimum sensitivity structures. Two numerical examples have shown that the proposed sensitivity is in very good agreement with the actual output error variance, and that minimum new statistical sensitivity form has a very small degradation of the frequency characteristic due to coefficient quantization.
Appendix

Derivation of $X$ and $V$

Using (4) and (12), we obtain

$$X(k+1) = E[x(k+1)x'(k+1)] = E[\{Ax(k) + bu(k)\} \cdot \{Ax(k) + bu(k)\}'] = AX(k)A' + bb' \quad (37)$$

$$V(k+1) = E[v(k+1)v'(k+1)] = E[\{Av(k) + \Delta A(k)x(k) + \Delta A(k)v(k) + \Delta b(k)u(k)\} \cdot \{Av(k) + \Delta A(k)x(k) + \Delta A(k)v(k) + \Delta b(k)u(k)\}'] = AV(k)A' + \sigma^2 tr[X(k)]I + \sigma^2 tr[V(k)]I + \sigma^2 I \quad (38)$$

In the steady state, letting

$$X = E[xx'] \quad (39)$$

$$V = E[vv'] \quad (40)$$

yields

$$X = AXA' + bb' \quad (41)$$

$$V = AVA' + \sigma^2 [tr[X] + tr[V] + 1]I \quad (42)$$

REFERENCES


