DESIGN OF VARIABLE DIGITAL FILTERS BASED ON STATE-SPACE REALIZATIONS

Hisashi MATSUKAWA and Masayuki KAWAMATA

Department of Electronic Engineering, Graduate School of Engineering
Tohoku University
Aoba-yama 05, Sendai, 980-8579, Japan
Phone:+81-22-217-7095, Fax:+81-22-263-9169,
Email:matukawa@mk.ecei.tohoku.ac.jp

ABSTRACT
This paper proposes a design method of variable IIR digital filters based on balanced realizations and minimum round-off noise realizations of digital filters. Highly accurate variable digital filters are easily derived by the proposed method. The coefficient matrices of both realizations of second-order digital filters are obtained directly from prototype realizations. The filter coefficients of variable digital filters can be obtained by frequency transformations to the realizations. The filter coefficients are presented as truncated Taylor series for the purpose of reducing a number of calculations to tune the coefficients. However the proposed filters have highly accurate variable characteristics against the coefficient truncation since balanced realizations and minimum round-off noise realizations have very low coefficient sensitivities, which are invariant under the frequency transformations. Numerical examples show the effectiveness of the variable digital filters designed by the proposed method.

1. INTRODUCTION

Variable digital filters are frequency selective filters with tunable frequency characteristics. They are often used in many signal processing applications, such as telecommunications, digital audio equipment and adaptive systems. Many structures and design methods of variable digital filters have already been proposed. Most of them are classified and compared by Stoyanov and Kawamata [1].

Variable digital filters are usually designed by employing the frequency transformations of Constantinides [2]. The transformations are based on the substitution:

$$ z^{-1} \rightarrow T(z) $$

where $T(z)$ is a first- or second-order all-pass transfer function. Applying the transformations to a prototype low-pass filter with the transfer function $H_p(z)$ and the cutoff frequency $\omega_{cp}$, a new transfer function $H(z)$ can be obtained as

$$ H(z) = H_p(z)|_{z^{-1}=T(z)}. $$

For low-pass to low-pass transformation, $T(z)$ is given by

$$ T(z) = \frac{z^{-1} - \xi}{1 - \xi z^{-1}} $$

$$ \xi = \frac{\sin(\omega_{cp} - \omega_c)}{\sin(\omega_{cp} + \omega_c)} $$

where $\omega_c$ is the cutoff frequency of the desired low-pass digital filter.

After applying the low-pass to low-pass transformation, the transfer function $H(z)$ with variable characteristics can be obtained. The coefficients of $H(z)$ are complicated functions of the variable parameter $\xi$. Therefore, they require a huge number of calculations to tune the coefficients. Linear approximation is applied to the coefficients in order to reduce the number of the calculations. This approximation is based on the Taylor series expansion with respect to $\xi$. The coefficients of the transfer function $H(z)$ are expanded into Taylor series, of which higher-order terms are truncated under the assumption $|\xi| \ll 1$. However, the linear approximation causes magnitude degradation. This degradation can be decreased by adopting low coefficient sensitivity structures.

In this paper, we employ balanced realizations and minimum round-off noise realizations to design variable digital filters. Both realizations are known as very low sensitivity structures. Kawamata, Iwatsuki and Higuchi [3] proved that balanced realizations have minimum statistical sensitivities. On the other hand, minimum round-off noise realizations investigated by Mullis and Roberts [4, 5], Hwang [6] and Barnes [7] have minimum statistical sensitivities under the $l_2$ norm scaling constraint [8]. Highly accurate variable digital filters can be derived from both realizations.

2. REVIEW OF BALANCED REALIZATIONS OF DIGITAL FILTERS

For the purpose of the following discussions, we ex-
expressed by the following state equations:

\[ y(n) = c^T x(n) + d^T u(n) \]

where \( x(n) \) is the \( N \)-th order state vector, \( u(n) \) is the scalar input sequence, \( y(n) \) is the scalar output sequence, and \( A, b, c \) and \( d \) are the \( N \times N, N \times 1, 1 \times N \) and \( 1 \times 1 \) real coefficient matrices, respectively. Figure 1 shows the signal flowgraph representation of the second-order state-space realization. The transfer function \( H(z) \) can be described as

\[ H(z) = c(zI - A)^{-1} b + d. \]

The covariance matrix \( K \) and the noise matrix \( W \) are defined as the solutions of the following Lyapunov equations:

\[ K = AKA^T + bb^T \]
\[ W = A^TW A + c^T c. \]

The matrices \( K \) and \( W \) depend on the filter structure.

The transfer function \( H(z) \) is invariant under nonsingular transformation matrices \( T \) of the state, i.e., if \( x' = T^{-1} x \), then the new realization \( (T^{-1} AT, T^{-1} b, cT, d) \) is an equivalent realization to \( (A, b, c, d) \) of the transfer function \( H(z) \). The covariance and noise matrices of the realization \( (T^{-1} AT, T^{-1} b, cT, d) \) are given by \( T^{-1} KT^{-T} \) and \( T^TW \), respectively.

Balanced realizations can also be derived by the suitable equivalent transformations. Balanced realizations are digital filters which have the following covariance and noise matrices:

\[ K' = W' = \text{diag}(\theta_1, \theta_2, \cdots, \theta_N) \]

where \( \theta_n (n = 1, 2, \cdots, N) \) are the second-order modes, which are the square roots of eigenvalues of the matrix product \( K'W' \). The second-order modes are invariant under the frequency transformations. Therefore, \( K' \) and \( W' \) are also invariant under the frequency transformations.

3. DESIGN OF BALANCED REALIZATIONS OF SECOND-ORDER DIGITAL FILTERS

3.1. Eigenvector expansions of coefficient matrices

Balanced realizations of second-order digital filters can be obtained directly from the transfer function by the design method proposed in this section. We consider a second-order digital filter with the transfer function

\[ H_p(z) = \frac{\alpha_p}{z - \lambda_p} + \frac{\alpha_p^*}{z - \lambda_p^*} + d_p \]

where

\[ \lambda_p = \sigma_p + j\omega_p \]
\[ \alpha_p = \alpha_{p^*} + j\alpha_{p^*} \]

The coefficient matrices of the filter are given by

\[ A_p = \begin{bmatrix} a_{p_{11}} & a_{p_{12}} \\ a_{p_{21}} & a_{p_{22}} \end{bmatrix} \]
\[ b_p = \begin{bmatrix} b_{p_1} \\ b_{p_2} \end{bmatrix} \]
\[ c_p = \begin{bmatrix} c_{p_1} & c_{p_2} \end{bmatrix} \]

In addition the second-order modes are expressed by Barnes [7] as

\[ \theta_1, \theta_2 = \sqrt{P^2 - Q^2} \pm R \]

where

\[ P = \frac{|\alpha_p|}{1 - |\lambda_p|^2} \]
\[ R + iQ = \frac{\alpha_p}{1 - \lambda_p^2} \]

From Eqs. (8) – (10), the constraints on the coefficient matrices \( A, b \) and \( c \) can be obtained as

\[ -a_{12} = a_{21} \]
\[ b_1 = c_1 \]
\[ b_2 = -c_2. \]

Under the constraint given by Eq. (20), the matrix \( A \) can be assumed as

\[ A = \begin{bmatrix} \sigma_p - i\omega_p & \sqrt{1 + \nu^2}\omega_p \\ -\sqrt{1 + \nu^2}\omega_p & \sigma_p + i\omega_p \end{bmatrix} \]
where $\nu$ is a parameter determined by the constraints on the covariance matrix and the noise matrix.

Then we employ the eigenvector expansions of coefficient matrices [7]. An eigenvector of the coefficient matrix $A$ is given by

$$\phi = \begin{bmatrix} 1 \\ \nu + j \end{bmatrix}$$

(24)

and the associated row eigenvector is given by

$$\psi = \frac{j}{2} \begin{bmatrix} \nu - j - \sqrt{1 + \nu^2} \end{bmatrix}.$$  

(25)

The row eigenvector $\psi$ is normalized so that

$$\psi \phi = \psi^* \phi^* = 1.$$  

(26)

The coefficient matrices $A$, $b$ and $c$ can be expanded on the eigenvector and the row eigenvector as follows:

$$A = \lambda \phi \phi + \lambda^* \phi^* \psi^*$$  

(27)

$$b = \beta \phi + \beta^* \phi^*$$  

(28)

$$c = \gamma \psi + \gamma^* \psi^*$$  

(29)

where

$$\beta = \beta_r + j \beta_i$$  

(30)

$$\gamma = \gamma_r + j \gamma_i.$$  

(31)

The coefficient matrices $b$ and $c$ can be denoted as

$$b = 2 \begin{bmatrix} \beta_r \\ \nu \beta_i - \beta_i \end{bmatrix}$$

(32)

$$c = \begin{bmatrix} \gamma_r - \nu \gamma_i \\ \gamma_i \sqrt{1 + \nu^2} \end{bmatrix}.$$  

(33)

The elements of the covariance matrix $K$ are given by

$$k_{11} = 2 \{ K_1 + \text{Re}(K_2) \}$$  

(37)

$$k_{12} = k_{21} = \frac{2}{\sqrt{1 + \nu^2}} [ \nu K_1 + \text{Re} \{ K_2 (\nu + j) \} ]$$  

(38)

$$k_{22} = 2 \left[ K_1 + \frac{1}{1 + \nu^2} \text{Re} \{ K_2 (\nu + j)^2 \} \right]$$  

(39)

where

$$K_1 = \frac{|\beta|^2}{1 - |\lambda|^2}$$  

(40)

$$K_2 = \frac{|\beta|^2}{1 - \lambda^2}.$$  

(41)

The elements of the noise matrix $W$ are given by

$$w_{11} = \frac{1 + \nu^2}{2} \{ W_1 + \text{Re} \{ W_2 (1 + j \nu)^2 \} \}$$  

(42)

$$w_{12} = w_{21} = \frac{\sqrt{1 + \nu^2}}{2} \left[ -\nu W_1 + \text{Re} \{ W_2 (\nu - j) \} \right]$$  

(43)

$$w_{22} = \frac{1 + \nu^2}{2} \{ W_1 - \text{Re} \{ W_2 \} \}$$  

(44)

where

$$W_1 = \frac{|\gamma|^2}{1 - |\lambda|^2}$$  

(45)

$$W_2 = \frac{|\gamma|^2}{1 - \lambda^2}.$$  

(46)

Under the constraints given by Eqs. (10), (35) and (36), the following equations can be obtained:

$$K_1 = \frac{\nu^2 + 1}{2} \sqrt{P^2 - Q^2}$$  

(47)

$$W_1 = 2 \sqrt{P^2 - Q^2}.$$  

(48)

From Eqs. (34), (47) and (48), we find that

$$\nu = \frac{\kappa - \kappa^{-1}}{2} = \frac{Q}{\sqrt{P^2 - Q^2}}$$  

(49)

where

$$\kappa = \sqrt{\frac{P + Q}{P - Q}}.$$  

(50)

Thus, the parameter $\nu$ is determined.

Therefore, the coefficient matrices of the second-order balanced realization are derived as

$$A_p = \begin{bmatrix} \sigma_p - \frac{\kappa - \kappa^{-1}}{2} \omega_p & \frac{\kappa + \kappa^{-1}}{2} \omega_p \\ -\frac{\kappa + \kappa^{-1}}{2} \omega_p & \sigma_p + \frac{\kappa - \kappa^{-1}}{2} \omega_p \end{bmatrix}$$  

(51)

$$b_p = \begin{bmatrix} \mu p_1 + \mu p_2 \\ \mu p_1 - \mu p_2 \end{bmatrix}$$  

(52)

$$c_p = \begin{bmatrix} b p_1 \\ -b p_2 \end{bmatrix}.$$  

(53)
where $\mu_1$ and $\mu_2$ are described as follows:

$$ \mu_1 = \sqrt{\frac{\kappa(|\alpha| - \alpha_{p1})}{2}} \quad (54) $$

$$ \mu_2 = \sqrt{\frac{|\alpha| + \alpha_{p1}}{2\kappa} \cdot \text{sign}(\alpha_{p1})} \quad (55) $$

From the second-order balanced realization given by Eqs. (51) – (55), the second-order minimum round-off noise realization can be obtained by the equivalent transformations. One of the equivalent transformation matrices $T$ is described as

$$ T = \sqrt{\frac{\kappa(P - Q)}{2}} \left[ \begin{array}{ccc} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] \quad (56) $$

This realization ($T^{-1}AT, T^{-1}b, c,b, d$) coincides with the second-order minimum round-off noise realization proposed by Barnes [7].

### 4. A NEW DESIGN METHOD OF VARIABLE LOW-PASS FILTERS

We employ $H_p(z)$ given by Eqs. (11) – (13) as a second-order prototype transfer function. Applying the low-pass to low-pass transformation to $H_p(z)$ yields a new transfer function $H(z)$:

$$ H(z) = \frac{\alpha}{z - \lambda} + \frac{\alpha^*}{z - \lambda^*} + d \quad (57) $$

where

$$ \alpha = \frac{1 - \xi^2}{(1 + \lambda \xi^2)\alpha_p} \quad (58) $$

$$ \lambda = \frac{\lambda \xi^2}{1 + \lambda \xi^2} \quad (59) $$

$$ d = d_p - 2\alpha_{p1}\xi + \frac{(\alpha_{p1} + \alpha_{p2}\omega_p)\xi^2}{1 + 2\sigma_p\xi + |\lambda|^2\xi^2} \quad (60) $$

The balanced realization of the transfer function can be obtained by the method explained in the previous section. Then the linear approximation is applied to all elements of the coefficient matrices. Therefore the coefficient matrices of the second-order variable low-pass digital filter are given by

$$ A = \begin{bmatrix} a_{p11} + \rho_1\xi & a_{p12}(1 - 2\sigma\xi) \\ -a_{p12}(1 - 2\sigma\xi) & a_{p22} + \rho_2 \xi \end{bmatrix} \quad (61) $$

$$ b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} |\mu_1| + |\mu_2| \cdot \text{sign}(\eta) \\ |\mu_1| - |\mu_2| \cdot \text{sign}(\eta) \end{bmatrix} \quad (62) $$

$$ c = \begin{bmatrix} b_1 \\ -b_2 \end{bmatrix} \quad (63) $$

$$ d = d_p - 2\alpha_{p1}\xi \quad (64) $$

### Table 1: The numbers of calculations to tune the coefficients of second-order variable low-pass filters

<table>
<thead>
<tr>
<th>Filter structure</th>
<th>Number of multiplications</th>
<th>Number of additions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Balanced realization</td>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td>MRN realization</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>Direct form II</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>Parallel all-pass structure</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>Lattice form</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

“MRN realization” denotes the minimum round-off noise realization.

where

$$ \rho_1 = |\lambda|^2 + 1 - 2\sigma_{p11}\sigma_p \quad (65) $$

$$ \rho_2 = |\lambda|^2 + 1 - 2\sigma_{p22}\sigma_p \quad (66) $$

$$ \mu_1 = \mu_{p1} - (\mu_{p1}\sigma_p - \mu_{p2}\omega_p)\xi \quad (67) $$

$$ \mu_2 = \mu_{p2} - (\mu_{p2}\sigma_p + \mu_{p1}\omega_p)\xi \quad (68) $$

$$ \eta = \alpha_{p1} + 2(\alpha_{p1}\sigma_p + \alpha_{p2}\omega_p)\xi + \{\alpha_{p1}(\sigma^2 - \omega_p^2) + 2\alpha_{p2}\sigma_p\omega_p\}\xi^2 \quad (69) $$

The second-order variable low-pass digital filter based on minimum round-off noise realizations can also be derived by the same equivalent transformation of which the matrix $T$ is given by Eq. (56). The coefficient matrices of the variable digital filter are expressed as ($T^{-1}AT, T^{-1}b, cT, d$), where $A, b, c$ and $d$ are given by Eqs. (61) – (69).

The higher-order variable low-pass filters can also be obtained by employing the proposed method. We design an $N$-th order variable low-pass filter where $N > 2$ as the parallel structure of first- and second-order sub-filters which are designed by the proposed method.

### 5. NUMBERS OF CALCULATIONS TO TUNE FILTER COEFFICIENTS

In this section, we compare the numbers of calculations to tune the coefficients of the second-order variable low-pass filters designed from the proposed realizations, direct form II, parallel all-pass structure [9] and lattice form [10]. Table 1 shows the numbers of the calculations. The numbers of the calculations of the proposed filters do not contain the number of the calculation of $\eta$ given by Eq. (69). This is because the calculation of $\eta$ is not necessary in many cases since the sign of $\eta$ does not change under the assumption $|\xi| \ll 1$.

The calculations of the proposed filters are reduced by the linear approximation. However, the variable filters designed by the proposed method need more calculations to
tune the coefficients than the other filters since the proposed filters have more coefficients than the others.

6. NUMERICAL EXAMPLES

We design the variable digital filters based on balanced realizations and minimum round-off noise realizations, and compare the proposed filters with the variable filters which are designed from the parallel structure of direct form II, the parallel all-pass structure [9] and the cascade structure of lattice form [10]. We adopt a sixth-order Butterworth low-pass filter with the cut-off frequency \(0.1\pi\) as the prototype filter.

Figure 2(a) shows the magnitude responses of the proposed filter based on balanced realizations when the parameter \(\xi\) varies from \(-0.2\) to \(0.2\). On the other hand, Figures 2(b), 2(c) and 2(d) show the magnitude responses of the other three variable filters, respectively. It is confirmed that the variable low-pass filter based on balanced realizations has more accurate magnitude responses than the other filters. The magnitude degradation of the parallel structure of direct form II is caused considerably in the case of \(\xi = \pm0.2\). The magnitude responses of the parallel all-pass structure deviate in the stopband due to the high stop-band sensitivity of the structure. The magnitude responses of the cascade structure of lattice form are completely destroyed in all case of \(\xi\). In addition the variable low-pass filter based on minimum round-off noise realizations has the magnitude responses which are as accurate as the responses shown in Figure 2(a).

Figure 3 plots the statistical sensitivities of the three different variable low-pass filters as functions of the variable
parameter $\xi$. The statistical sensitivity, proposed by Kawamata et al. [3], is defined as
\[
S = \frac{\text{tr}(K) + 1}{\text{tr}(W) + 1}.
\]  (70)

As shown in Figure 3, the variable filters designed by the proposed method have very low statistical sensitivities. The parallel structure of direct form II has high sensitivity which increases with $\xi$. The sensitivities of the parallel structure of balanced realizations and the parallel structure of minimum round-off noise realizations also depend on $\xi$, but the variations of the sensitivities are smaller than that of the parallel structure of direct form II. It is confirmed that the variable digital filters designed by the proposed method preserve the invariance of the statistical sensitivities under the frequency transformation of balanced realizations and minimum round-off noise realizations.

7. CONCLUSIONS

We have proposed the design method of variable digital filters based on balanced realizations and minimum round-off noise realizations. The variable digital filters designed by the proposed method have very low coefficient sensitivities, and preserve the invariance of the sensitivity under the frequency transformations of the prototype realizations. Numerical examples show the effectiveness of the proposed method.

REFERENCES


