Design of Variable Digital Filters Based on State-Space Realizations

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SUMMARY This paper proposes a design method of variable IIR digital filters based on balanced realizations and minimum round-off noise realizations of digital filters. Highly accurate variable digital filters are easily derived by the proposed method. The coefficient matrices of both realizations of second-order digital filters are obtained directly from prototype realizations. The filter coefficients of variable digital filters can be obtained by frequency transformations to the realizations. The filter coefficients are presented as truncated Taylor series to tune the coefficients. However, the proposed filters have highly accurate variable characteristics against the coefficient truncation since balanced realizations and minimum round-off noise realizations have very low coefficient sensitivities, which are invariant under the frequency transformations. Moreover, the dynamic ranges of the proposed filters are almost constant against the frequency transformations. Numerical examples show the effectiveness of the variable digital filters designed by the proposed method.

key words: variable digital filter, balanced realization, minimum round-off noise realization, statistical sensitivity

1. Introduction

Variable digital filters are frequency selective filters with tunable frequency characteristics. They are often used in many signal processing applications, such as telecommunications, digital audio equipment and adaptive systems. Many structures and design methods of variable digital filters have already been proposed. Most of them are classified and compared by Stoyanov and Kawamata [1].

Variable digital filters are usually designed by employing the frequency transformations of Constantinides [2]. The transformations are based on the substitution:

\[ z^{-1} \rightarrow T(z) \]  

where \( T(z) \) is a first- or second-order all-pass transfer function. Applying the transformations to a prototype low-pass (LP) filter with the transfer function \( H_p(z) \) and the cutoff frequency \( \omega_c \), a new transfer function \( H(z) \) can be obtained as

\[ H(z) = H_p(z)|_{z^{-1}=T(z)}. \]  

For LP to LP transformation, \( T(z) \) is given by the following first-order all-pass function:

\[ T(z) = \frac{z^{-1} - \xi}{1 - \xi z^{-1}} \]

\[ \xi = \frac{\sin \left( \frac{\omega_{lp} - \omega_c}{2} \right)}{\sin \left( \frac{\omega_{lp} + \omega_c}{2} \right)} \]

where \( \omega_c \) is the cutoff frequency of the desired LP digital filter.

After applying the LP to LP transformation, the transfer function \( H(z) \) with variable characteristics can be obtained. The coefficients of \( H(z) \) are complicated functions of the variable parameter \( \xi \). Therefore, they require a huge number of calculations to tune the coefficients. Linear approximation is applied to the coefficients in order to reduce the number of the calculations. This approximation is based on the Taylor series expansion with respect to \( \xi \). The coefficients of the transfer function \( H(z) \) are expanded into Taylor series, of which higher-order terms are truncated under the assumption \( |\xi| \ll 1 \). However, the linear approximation causes magnitude degradation. This degradation can be decreased by adopting low coefficient sensitivity structures.

In this paper, we propose to employ balanced realizations and minimum round-off noise (MRN) realizations to design variable digital filters. Both realizations are known as very low sensitivity structures. Kawamata, Iwatsuki and Higuchi proved that balanced realizations have minimum statistical sensitivities [3], [4]. On the other hand, MRN realizations investigated by Mullis and Roberts [5], [6], Hwang [7] and Barnes [8] have minimum statistical sensitivities under the \( l_2 \) norm scaling constraint [9]. Both realizations of second-order digital filters can be derived in closed form from the transfer function by the synthesis methods proposed in this paper, and they are related by an equivalent transformation \( T \) of simple scaling and rotation. Highly accurate variable digital filters can be derived from these realizations.

This paper is organized as follows: Section 2 discusses state-space realizations of digital filters. Sections 3 and 4 give synthesis methods of second-order
balanced realizations and second-order MRN realizations. Section 5 proposes a new design method of variable LP digital filters. Section 6 discusses numbers of calculations to tune filter coefficients of the proposed filters. Section 7 compares the characteristics of the proposed filters with those of the other conventional filters.

2. Review of State-Space Realizations of Digital Filters

For the purpose of the following discussions, we explain state-space realizations of digital filters. For a given \( H(z) \), the digital filter can be expressed by the following state equations:

\[
x(n+1) = Ax(n) + bu(n) \quad (5)
\]

\[
y(n) = cx(n) + du(n) \quad (6)
\]

where \( x(n) \) is the \( N \)th-order state vector, \( u(n) \) is the scalar input sequence, \( y(n) \) is the scalar output sequence, and \( A, b, c \) and \( d \) are the \( N \times N, N \times 1, 1 \times N \) and \( 1 \times 1 \) real coefficient matrices, respectively. Figure 1 shows the signal flowgraph representation of the second-order state-space realization. The transfer function \( H(z) \) can be described as

\[
H(z) = c(zI - A)b + d. \quad (7)
\]

The covariance matrix \( K \) and the noise matrix \( W \) are defined as the solutions of the following Lyapunov equations:

\[
K = AKA^T + bb^T \quad (8)
\]

\[
W = A^TWA + c'c. \quad (9)
\]

The matrices \( K \) and \( W \) depend on the filter structure. The diagonal elements of \( K \) mean the dynamic ranges of state variables. The statistical properties of the digital filters, proposed by Kawamata et al. [3], [4], is defined as

\[
S = \{\text{tr}(K) + 1\} \{\text{tr}(W) + 1\}. \quad (10)
\]

The transfer function \( H(z) \) is invariant under nonsingular transformation matrices \( T \) of the state, i.e., if \( x' = T^{-1}x \), then the new realization \( (T^{-1}AT, T^{-1}b, cT, d) \) is an equivalent realization to \( (A, b, c, d) \) of the transfer function \( H(z) \). The covariance and noise matrices of the realization \( (T^{-1}AT, T^{-1}b, cT, d) \) are given by \( T^{-1}KT^{-1} \) and \( T^{-1}WT \), respectively.

Balanced realizations can also be derived by the suitable equivalent transformations. Balanced realizations are digital filters which have the following covariance and noise matrices:

\[
K' = W' = \text{diag}(\theta_1, \theta_2, \cdots, \theta_N) \quad (11)
\]

where \( \theta_n \) are the second-order modes, which are the square roots of eigenvalues of the matrix product \( K'W' \). The second-order modes are invariant under the frequency transformations. Therefore, \( K' \) and \( W' \) are also invariant under the frequency transformations.

3. Design of Balanced Realizations of Second-Order Digital Filters

In this section, balanced realizations of second-order digital filters are derived directly from the transfer function by eigenvector expansions of coefficient matrices. The eigenvector expansions were applied to the design of MRN realizations of second-order digital filters [8]. Balanced realizations of second-order digital filters can be derived by applying this method.

3.1 Eigenvector Expansions of Coefficient Matrices

We consider a second-order digital filter with the transfer function

\[
H_p(z) = \frac{\alpha_p}{z - \lambda_p} + \frac{\alpha_p^*}{z - \lambda_p^*} + d_p \quad (12)
\]

where

\[
\lambda_p = \sigma_p + j\omega_p \quad (13)
\]

\[
\alpha_p = \alpha_{pr} + j\alpha_{pi}. \quad (14)
\]

The coefficient matrices of the filter are given by

\[
A_{60} = \begin{bmatrix} a_{6011} & a_{6012} \\ a_{6021} & a_{6022} \end{bmatrix}
\]

\[
b_{60} = \begin{bmatrix} b_{601} \\ b_{602} \end{bmatrix}
\]

\[
c_{60} = \begin{bmatrix} c_{601} & c_{602} \end{bmatrix}
\]

\[
d_{60} = d_p. \quad (15)
\]
In addition the second-order modes are expressed by Barnes [8] as
\[ \theta_1, \theta_2 = \sqrt{P^2 - Q^2} \pm R \] (16)
where
\[ P = \frac{|\alpha_p|}{1 - |\lambda_p|^2} \] (17)
\[ R + iQ = \frac{\alpha_p}{1 - \lambda_p^2}. \] (18)
The parameters \( P, Q \) and \( R \) are invariant under the frequency transformations [8].

From Eqs. (8), (9) and (11), the constraints on the coefficient matrices \( A_{b_0}, b_{b_0} \) and \( c_{b_0} \) can be obtained as
\[ a_{b_012} = a_{b_021} \] (19)
\[ b_{b_01} = c_{b_01} \] (20)
\[ b_{b_02} = -c_{b_02}. \] (21)
Under the constraint given by Eq. (19), the matrix \( A_{b_0} \) can be assumed as
\[ A_{b_0} = \begin{bmatrix} \sigma_p - \nu \omega_p & \sqrt{1 + \nu^2} \omega_p \\ -\sqrt{1 + \nu^2} \omega_p & \sigma_p + \nu \omega_p \end{bmatrix} \] (22)
where \( \nu \) is a parameter determined by the constraints on the covariance matrix and the noise matrix.

Then we employ the eigenvector expansions of coefficient matrices [8]. An eigenvector of the coefficient matrix \( A_{b_0} \) is given by
\[ \phi = \begin{bmatrix} \frac{1}{\sqrt{1 + \nu^2}} \\ \nu + j \frac{1}{\sqrt{1 + \nu^2}} \end{bmatrix} \] (23)
and the associated row eigenvector is given by
\[ \psi = \frac{j}{2} \left( \nu - j \sqrt{1 + \nu^2} \right). \] (24)
The row eigenvector \( \psi \) is normalized so that
\[ \psi \phi = \psi^* \phi^* = 1. \] (25)
The coefficient matrices \( A_{b_0}, b_{b_0} \) and \( c_{b_0} \) can be expanded on the eigenvector and the row eigenvector as follows:
\[ A_{b_0} = \lambda_p \phi \psi + \lambda_p^* \phi^* \psi^* \] (26)
\[ b_{b_0} = \beta \phi + \beta^* \phi^* \] (27)
\[ c_{b_0} = \gamma \psi + \gamma^* \psi^* \] (28)
where
\[ \beta = \beta_r + j \beta_i \] (29)
\[ \gamma = \gamma_r + j \gamma_i. \] (30)
The coefficient matrices \( b_{b_0} \) and \( c_{b_0} \) can be denoted as
\[ b_{b_0} = 2 \begin{bmatrix} \beta_r \\ \nu \beta_r - \beta_i \sqrt{1 + \nu^2} \end{bmatrix} \] (31)
\[ c_{b_0} = \begin{bmatrix} \gamma_r - \nu \gamma_i \\ \gamma_i \sqrt{1 + \nu^2} \end{bmatrix}. \] (32)

### 3.2 Design Procedure of Balanced Realizations

The parameters \( \beta \) and \( \gamma \) must satisfy the constraints given by Eqs. (12), (20) and (21). In order to satisfy Eq. (12), \( \beta \) and \( \gamma \) must be chosen as follows:
\[ \beta \gamma = \alpha. \] (33)
From Eqs. (20) and (21), \( \gamma \) must be expressed as
\[ \gamma_r = \frac{2}{1 + \nu^2} (\beta_r + \nu \beta_i) \] (34)
\[ \gamma_i = \frac{2}{1 + \nu^2} (-\nu \beta_r + \beta_i). \] (35)
The covariance and noise matrices are represented by the assumed coefficient matrices given by Eqs. (22), (31) and (32). The elements of the covariance matrix \( K_{b_0} \) are given by
\[ k_{b_011} = 2 \{ K_1 + \text{Re} (K_2) \} \] (36)
\[ k_{b_012} = k_{b_021} = \frac{2}{\sqrt{1 + \nu^2}} \nu K_1 + \text{Re} \{ K_2 (\nu + j) \} \] (37)
\[ k_{b_022} = 2 \left[ K_1 + \frac{1}{1 + \nu^2} \text{Re} \{ K_2 (\nu + j)^2 \} \right] \] (38)
where
\[ K_1 = \frac{|\beta|}{1 - |\lambda_p|^2} \] (39)
\[ K_2 = \frac{\beta^2}{1 - \lambda_p^2}. \] (40)
The elements of the noise matrix \( W_{b_0} \) are given by
\[ w_{b_011} = \frac{1 + \nu^2}{2} W_1 + \text{Re} \{ W_2 (1 + j \nu)^2 \} \] (41)
\[ w_{b_012} = w_{b_021} = \frac{\sqrt{1 + \nu^2}}{2} \nu W_1 + \text{Re} \{ W_2 (\nu - j) \} \] (42)
\[ w_{b_022} = \frac{1 + \nu^2}{2} \{ W_1 - \text{Re} (W_2) \} \] (43)
where
\[ W_1 = \frac{|\gamma|^2}{1 - |\lambda_p|^2} \] (44)
\[ W_2 = \frac{\gamma^2}{1 - \lambda_p^2}. \] (45)
Under the constraints given by Eqs. (11), (34) and (35), the following equations can be obtained:

\[ K_1 = \frac{\nu^2 + 1}{2} \sqrt{P^2 - Q^2} \quad (46) \]
\[ W_1 = 2 \sqrt{P^2 - Q^2}. \quad (47) \]

From Eqs. (33), (46) and (47), we find that

\[ \nu = \frac{\kappa - \kappa^{-1}}{2} = \frac{Q}{\sqrt{P^2 - Q^2}} \quad (48) \]

where

\[ \kappa = \sqrt{\frac{P + Q}{P - Q}} \quad (49) \]

Thus, the parameter \( \nu \) is determined.

Therefore, the coefficient matrices of the second-order balanced realization are given as

\[
A_{b0} = \begin{bmatrix}
\sigma _p - \frac{\kappa - \kappa^{-1}}{2} \omega _p & \frac{\kappa + \kappa^{-1}}{2} \omega _p \\
-\frac{\kappa + \kappa^{-1}}{2} \omega _p & \sigma _p + \frac{\kappa - \kappa^{-1}}{2} \omega _p
\end{bmatrix}
\]

\[
b_{b0} = \begin{bmatrix} \mu _{p1} + \mu _{p2} \\ \mu _1 - \mu _{p2} \end{bmatrix}
\]

\[
c_{b0} = \begin{bmatrix} b_{b01} & -b_{b02} \end{bmatrix}
\]

\[
d_{b0} = d_p
\]

where \( \mu _{p1} \) and \( \mu _{p2} \) are described as follows:

\[ \mu _{p1} = \sqrt{\frac{\kappa (|\alpha _p| - \alpha _{p_1})}{2}} \quad (51) \]
\[ \mu _{p2} = \sqrt{\frac{|\alpha _p| + \alpha _{p_1}}{2\kappa}} \cdot \text{sign}(\alpha _{p_r}). \quad (52) \]

### 3.3 Covariance and Noise Matrices and Statistical Sensitivity of Balanced Realizations

The covariance matrix and the noise matrix of the second-order balanced realization are given by

\[
K_{b0} = W_{b0} = \begin{bmatrix} \sqrt{P^2 - Q^2} + R & 0 \\ 0 & \sqrt{P^2 - Q^2} - R \end{bmatrix}. \quad (53)
\]

Thus, from Eqs. (10) and (53), the statistical sensitivity of the second-order balanced realization is given by

\[ S_{b0} = \left(2\sqrt{P^2 - Q^2} + 1\right)^2. \quad (54) \]

From Eqs. (53) and (54), it is confirmed that the statistical sensitivity and the dynamic ranges of the state variables of the second-order balanced realization are invariant under the frequency transformations, since \( P \), \( Q \) and \( R \) are invariant under the frequency transformations. Therefore, variable digital filters of which sensitivities and dynamic ranges are constant against the frequency transformations can be derived by adopting the second-order balanced realization.

### 4. Design of MRN Realizations of Second-Order Digital Filters

#### 4.1 Design Procedure of MRN Realizations

The MRN realizations and the balanced realizations of any order are related by the following simple equivalent transformation [10]:

\[ T = \rho \frac{1}{\sqrt{2}} U^T \quad (55) \]

where \( \rho \) is the mean of the second-order modes and \( U \) is an orthogonal matrix.

It is very difficult to find an explicit formula of \( T \) for higher-order case. But, for second-order case, we can find an explicit formula of \( T \) as follows. For the second-order case, the covariance matrix, the noise matrix and \( \rho \) of the MRN realization \( (A_{m0}, b_{m0}, c_{m0}, d_{m0}) \) are obtained from the covariance and noise matrices of the second-order balanced realization given in Eq. (53) as follows:

\[
K_{m0} = T^{-1} K_{b0} T^{-T} = \rho^{-1} U^{-T} K_{b0} U^{-1} \quad (56)
\]
\[
W_{m0} = T^T W_{b0} T = \rho U W_{b0} U^T \quad (57)
\]
\[
\rho = \sqrt{P^2 - Q^2}. \quad (58)
\]

The orthogonal matrix \( U \) can be assumed as

\[ U = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}. \quad (59) \]

The covariance matrix \( K_{m0} \) must satisfy the following \( l_2 \) norm constraint:

\[ k_{m011} = k_{m022} = 1 \quad (60) \]

where \( k_{m011} \) and \( k_{m022} \) are the diagonal elements of \( K_{m0} \). From Eqs. (56)–(60), we find \( x = \pm \frac{\pi}{4}, \pm \frac{3\pi}{4}. \) If \( x = \frac{\pi}{4}, \) \( U \) is given by

\[ U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}. \quad (61) \]

Thus, the second-order MRN realization can be obtained by the following equivalent transformation:

\[ T = (P^2 - Q^2)^{\frac{1}{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}. \quad (62) \]

Then, the second-order MRN realization \( (A_{m0}, b_{m0}, c_{m0}, d_{m0}) = T^{-1} A_{b0} T, T^{-1} b_{b0}, c_{b0} T, d_{b0} \) is obtained as follows:

\[
A_{m0} = \begin{bmatrix} a_{m011} & a_{m012} \\ a_{m021} & a_{m022} \end{bmatrix} = \begin{bmatrix} \frac{\sigma _p + \kappa \omega _p}{\sigma _p} \\ \frac{\sigma _p - \kappa \omega _p}{\sigma _p} \end{bmatrix}.
\]
This realization \( (A_{m0}, b_{m0}, c_{m0}, d_{m0}) \) can also be derived by the synthesis method proposed by Barnes [8], but the second-order balanced realization and the second-order MRN realization can be investigated uniformly by the relation given by Eq. (62). This is the reason the second-order MRN realization is derived from the second-order balanced realization by the equivalent transformation.

### 4.2 Covariance and Noise Matrices and Statistical Sensitivity of MRN Realizations

The covariance matrix, the noise matrix and the statistical sensitivity of the second-order MRN realization are given by

\[
K_{m0} = \begin{bmatrix} 1 & \lambda \alpha \sigma \omega \xi \\ -\lambda \alpha \sigma \omega \xi & \alpha \sigma \omega \xi^2 \end{bmatrix}
\]

\[
W_{m0} = (P^2 - Q^2) K_{m0}
\]

\[
S_{m0} = 3 \{ 2(P^2 - Q^2) + 1 \}
\]

From Eqs. (64)–(66), we find that the statistical sensitivity and the dynamic ranges of the state variables of the second-order MRN realization are also invariant under the frequency transformations. Therefore, variable digital filters of which sensitivities and dynamic ranges are constant against the frequency transformations can be derived by adopting the second-order MRN realization. Moreover, variable digital filters based on the second-order MRN realization satisfy the \( l_2 \) norm constraint against the frequency transformations.

### 5. A New Design Method of Variable LP Filters

#### 5.1 Variable LP Filters Based on Balanced Realizations

We employ \( H(z) \) given by Eqs. (12)–(14) as a second-order prototype transfer function. Applying the LP to LP transformation to \( H(z) \) yields a new transfer function \( \tilde{H}(z) \):

\[
\tilde{H}(z) = \frac{\alpha}{z - \lambda} + \frac{\alpha^*}{z - \lambda^*} + d
\]

where

\[
\alpha = \frac{1 - \xi^2}{(1 + \lambda \rho \xi)^2} \alpha_p
\]

\[
\lambda = \frac{\lambda \rho \xi}{1 + \lambda \rho \xi}
\]

\[
d = d_p - 2(\alpha_p \sigma_p + \alpha_p \omega_p \xi) \frac{\lambda^2 \xi^2}{1 + 2\sigma_p \xi + |\lambda|^2 \xi^2}
\]

The balanced realization of the transfer function \( H(z) \) can be obtained by the method explained in the previous section. For example, one of the coefficients can be obtained as follows:

\[
a_{b11} = \frac{a_{b011} + (|\lambda|^2 + 1) \xi + a_{b022} \xi^2}{1 + 2\sigma_p \xi + |\lambda|^2 \xi^2}
\]

Next, the coefficients of the transfer function \( H(z) \) are expanded into Taylor series with respect to \( \xi \). The coefficient \( a_{b11} \) is expanded into Taylor series as the following equation:

\[
a_{b11} = \sum_{n=0}^{\infty} \frac{\partial^n a_{b11}}{\partial \xi^n} \Bigg|_{\xi=0} \cdot \xi^n
\]

Then the higher-order \( (n \geq 2) \) terms of the Taylor series are truncated under the assumption \( |\xi| \ll 1 \). Therefore the coefficient matrices of the second-order variable LP digital filter are given by

\[
A_b = \begin{bmatrix} a_{b011} + \rho_{b1} \xi & a_{b012} \xi \\ -a_{b012} \xi & a_{b022} + \rho_{b2} \xi \end{bmatrix}
\]

\[
b_b = \begin{bmatrix} b_{b1} \\ b_{b2} \end{bmatrix} = \begin{bmatrix} |\mu_1| + |\mu_2|, \text{sgn}(\eta) \\ |\mu_1| - |\mu_2|, \text{sgn}(\eta) \end{bmatrix}
\]

\[
c_b = \begin{bmatrix} b_{b1} \\ -b_{b2} \end{bmatrix}
\]

\[
d_b = d_p - 2 \alpha_p \sigma_p \xi
\]

where

\[
\rho_{b1} = |\lambda|^2 + 1 - 2a_{b011} \sigma_p
\]

\[
\rho_{b2} = |\lambda|^2 + 1 - 2a_{b022} \sigma_p
\]

\[
\mu_1 = \mu_{p1} - (\mu_{p1} \sigma_p + \mu_{p2} \omega_p) \xi
\]

\[
\mu_2 = \mu_{p2} - (\mu_{p2} \sigma_p + \mu_{p1} \omega_p) \xi
\]

\[
\eta = \alpha_p + 2(\alpha_p \sigma_p + \alpha_p \omega_p) \xi
\]

\[
+ (\alpha_p \sigma_p^2 - \omega_p^2) + 2\alpha_p \sigma_p \omega_p \xi^2.
\]

#### 5.2 Variable LP Filters Based on MRN Realizations

The second-order variable LP digital filter based on MRN realizations can also be derived by the same procedure. The coefficient matrices of the variable digital filter are expressed as
\\[ A_m = \begin{bmatrix} \sigma_p + \mu_m \xi & a_{m012}(1 - 2\sigma_p \xi) \\ a_{m021}(1 - 2\sigma_p \xi) & \sigma_p + \mu_m \xi \end{bmatrix} \]

\[ b_m = \begin{bmatrix} b_{m1} \\ b_{m2} \end{bmatrix} = \sqrt{2} \left[ \kappa(P - Q) \right] \begin{bmatrix} |\mu_1| \\ -|\mu_2| \cdot \text{sign}(\eta) \end{bmatrix} \]

\[ c_m = \begin{bmatrix} c_{m1} \\ c_{m2} \end{bmatrix} \]

\[ d_m = d_p - 2\alpha_p \xi \]

where

\[ \rho_m = 1 - \sigma_p^2 + \omega_p^2 \]

\[ c_{m1} = \frac{\alpha_p - \sigma_p + (|\alpha_p| - |\alpha_p|)\omega_p}{b_{m01}} \cdot \text{sign}(\eta) \]

\[ c_{m2} = \frac{\alpha_p - \sigma_p + (|\alpha_p| + |\alpha_p|)\omega_p}{b_{m02}} \]

5.3 Higher-Order Cases

The higher-order variable LP filters can also be obtained by employing the proposed method. We design an Nth-order variable LP filter where N > 2 as the parallel structure of first- and second-order sub-filters which are designed by the proposed method.

5.4 Stability Condition

A transfer function which is derived from the prototype by the LP to LP transformation is stable under the condition given by

\[ |\xi| < 1. \] (83)

However applying linear approximation to the coefficients, the possible range of \( \xi \) is narrower than the range given by Eq. (83).

The stability condition of the proposed variable LP filters is given by

\[ |\text{eig}(A_b)| = |\text{eig}(A_m)| < 1. \] (84)

From this condition, the possible range of \( \xi \) must be limited as follows:

\[ \max(\xi_1, -1) < \xi < \min(\xi_2, 1) \]

where \( \xi_1 \) and \( \xi_2 \) (\( \xi_1 < \xi_2 \)) are the solutions of the following equation

\[ \left\{ (1 + |\lambda_p|^2)^2 - 4\sigma_p^2 \right\} \xi^2 + 2\sigma_p (1 - |\lambda_p|^2) \xi + |\lambda_p|^2 - 1 = 0. \] (86)

### 6. Numbers of Calculations to Tune Filter Coefficients

In this section, we compare the numbers of calculations to tune the coefficients of the second-order variable LP filters designed from the proposed realizations, direct form II, parallel all-pass structure [11] and lattice form [12]. Table 1 shows the numbers of the calculations.

The numbers of the calculations of the proposed filters do not contain the number of the calculation of \( \eta \) given by Eq. (78). This is because the calculation of \( \eta \) is not necessary in many cases since the sign of \( \eta \) does not change under the assumption \( |\xi| \ll 1 \).

The calculations of the proposed filters are reduced by the linear approximation. However, the proposed filters need more calculations to tune the coefficients than the other filters since the proposed filters have more coefficients than the others.

### 7. Numerical Examples

7.1 Frequency Characteristics

We design the variable digital filters based on balanced realizations and MRN realizations, and compare the proposed filters with the variable filters which are designed from the parallel structure of direct form II, the parallel all-pass structure [11] and the cascade structure of lattice form [12]. We adopt a sixth-order Butterworth LP filter with the cut-off frequency 0.1\( \pi \) as the prototype filter.

Figures 2(a) and (b) show the magnitude responses of the proposed filters based on balanced realizations and MRN realizations when the parameter \( \xi \) varies from -0.2 to 0.2, respectively. On the other hand, Figs. 2(c), (d) and (e) show the magnitude responses of the other three variable filters, respectively. Figure 2(f) indicates the deviations between the ideal response and the magnitude responses, and Table 2 shows the peak deviations of these filters.

It is confirmed that the variable LP filter based on balanced realizations has more accurate magnitude responses than the other filters. The magnitude degradation of the parallel structure of direct form II is caused considerably in the case of \( \xi = \pm 0.2 \). The magnitude responses of the parallel all-pass structure deviate in

<table>
<thead>
<tr>
<th>Filter structure</th>
<th>Number of multiplications</th>
<th>Number of additions</th>
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<tbody>
<tr>
<td>Balanced realization</td>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td>MRN realization</td>
<td>8</td>
<td>8</td>
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<tr>
<td>Direct form II</td>
<td>5</td>
<td>5</td>
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<tr>
<td>Parallel all-pass structure</td>
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<td>6</td>
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<tr>
<td>Lattice form</td>
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</tbody>
</table>
(a) The parallel structure of second-order balanced realizations.
(b) The parallel structure of second-order MRN realizations.
(c) The parallel structure of second-order direct form II.
(d) The parallel all-pass structure.
(e) The cascade structure of second-order lattice form.
(f) The deviations between the ideal response and the magnitude responses.

Fig. 2 The magnitude responses of the sixth-order variable digital filters.

the stopband due to the high stopband sensitivity of the structure. The magnitude responses of the cascade structure of lattice form are completely destroyed in the all case of $\xi$. In addition the variable LP filter based on MRN realizations has the magnitude responses which are as accurate as the responses shown in Fig. 2(a).

7.2 Statistical Sensitivities

Figure 3 plots the statistical sensitivities of the three different variable LP filters as functions of the variable parameter $\xi$. As shown in Fig. 3, the variable filters designed by the proposed method have very low statistical
Table 2  The peak deviations of the sixth-order variable digital filters ($\xi = 0.2$).

<table>
<thead>
<tr>
<th>Filter structure</th>
<th>Peak deviation (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parallel balanced realization</td>
<td>$-14.13$</td>
</tr>
<tr>
<td>Parallel MRN realization</td>
<td>$-14.13$</td>
</tr>
<tr>
<td>Parallel direct form II</td>
<td>$-2.98$</td>
</tr>
<tr>
<td>Parallel all-pass structure</td>
<td>$-12.72$</td>
</tr>
<tr>
<td>Cascade lattice form</td>
<td>$-0.20$</td>
</tr>
</tbody>
</table>

Fig. 3  The statistical sensitivities of the six-order variable digital filters.

sensitivities, which are almost constant against the parameter $\xi$. The parallel structure of direct form II has high sensitivity which extremely increases with $\xi$. The sensitivities of the parallel structure of balanced realizations and the parallel structure of MRN realizations also depend on $\xi$, but the variations of the sensitivities are much smaller than that of the parallel structure of direct form II. It is confirmed that the variable digital filters designed by the proposed method preserve the invariance of the statistical sensitivities under the frequency transformations of balanced realizations and MRN realizations.

In addition Fig. 3 shows that the variable LP filter based on balanced realizations has a lower statistical sensitivity than the variable LP filter based on MRN realizations. The proposed method based on balanced realizations is superior to the proposed method based on MRN realizations in terms of the statistical sensitivity.

7.3 Dynamic Ranges

Figure 4 plots the means of the dynamic ranges of the three different variable LP filters as functions of the variable parameter $\xi$. In this figure, the dynamic ranges indicate the variances of the state variables. Figure 4 shows the parallel structure of direct form II has the dynamic ranges which extremely increase with $\xi$. The dynamic ranges of the parallel structure of balanced realizations and the parallel structure of MRN realizations also depend on $\xi$, but the variations of the dynamic ranges are much smaller than those of the parallel structure of direct form II. It is confirmed that the variable digital filters designed by the proposed method also preserve the invariance of the dynamic ranges under the frequency transformations of balanced realizations and MRN realizations.

In addition Fig. 4 shows that the variable LP filter based on MRN realizations satisfies the $l_2$ norm constraint against the frequency transformations. The variable digital filter based on balanced realization also has constant dynamic ranges, but does not satisfy the constraint. The proposed method based on MRN realizations is superior to the proposed method based on balanced realizations in terms of the dynamic ranges.

8. Conclusions

We have proposed the design method of variable digital filters based on balanced realizations and MRN realizations. The variable digital filters designed by the proposed method have very low coefficient sensitivities, and preserve the invariance of the sensitivity and the dynamic ranges of state variables under the frequency transformations of the prototype realizations. Numerical examples show that the proposed filters have wider possible ranges of the variable parameter $\xi$ than the conventional filters, and preserve the invariance under the frequency transformations of the prototype realizations.

References


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