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ON THE INVARIANCE OF SECOND-ORDER MODES UNDER FREQUENCY TRANSFORMATION IN 2-D SEPARABLE DENOMINATOR DIGITAL FILTERS

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ABSTRACT
This paper discusses the invariance of the second-order modes of 2-D separable denominator digital filters under frequency transformation. It is proved that the second-order modes of 2-D separable denominator digital filters are not invariant under all frequency transformations, but invariant under specific frequency transformations of which transfer functions are strictly proper.

1. INTRODUCTION
The second-order modes of 1-D and 2-D digital filters are defined as the square roots of the eigenvalues of the product of the controllability grammian and the observability grammian of the filters [1, 2, 3]. They play very important roles in model reduction problem and minimization problem of quantization effects of 1-D and 2-D digital filters. In Ref. [1], Mullis and Roberts proved the very important property that the second-order modes of 1-D digital filters are invariant under any frequency transformation. They revealed by using this property that the roundoff noise of the 1-D optimal realization of digital filters is constant under any frequency transformation and thus the minimum value of the roundoff noise is not dependent on the bandwidth of the frequency characteristics.

In the case of 2-D digital filters, however, to the authors’ best knowledge, there is no research on the invariance of the second-order modes. In this paper, restricting ourselves to the 2-D separable denominator case, we will discuss the invariance problem of 2-D digital filters in state-space expression, and prove that the second-order modes of 2-D separable denominator digital filters are not invariant under all frequency transformations, but invariant under specific frequency transformations of which transfer functions are strictly proper. We will further give some practical frequency transformations providing the invariance of the second-order modes.

2. PRELIMINARIES

2.1. State-space model
The input-output relation of an \((N_1, N_2)\)-th order 2-D separable denominator digital filter is represented by the following Roesser’s state-space equations [4, 5]:

\[
\begin{bmatrix}
    x^h(n_1 + 1, n_2) \\
    x^v(n_1, n_2 + 1)
\end{bmatrix} =
\begin{bmatrix}
    A_1 & 0 \\
    A_3 & A_4
\end{bmatrix}
\begin{bmatrix}
    x^h(n_1, n_2) \\
    x^v(n_1, n_2)
\end{bmatrix} +
\begin{bmatrix}
    b_1 \\
    b_2
\end{bmatrix} u(n_1, n_2)
\]

\[
y(n_1, n_2) =
\begin{bmatrix}
    c_1 & c_2
\end{bmatrix}
\begin{bmatrix}
    x^h(n_1, n_2) \\
    x^v(n_1, n_2)
\end{bmatrix} + du(n_1, n_2) \tag{1}
\]

where \(u(n_1, n_2)\) and \(y(n_1, n_2)\) are the scalar input and output of the filter, and \(x^h(n_1, n_2)\) and \(x^v(n_1, n_2)\) are the \(N_1 \times 1\) horizontal state vector and the \(N_2 \times 1\) vertical state vector, respectively. Matrices \(A_1, A_3, A_4, b_1, b_2, c_1, c_2,\) and \(d\) are real coefficient matrices with appropriate size. The block diagram of the 2-D digital filter is given in Fig. 1. The transfer function \(H(z_1, z_2)\) of the 2-D filter is given in
terms of the coefficient matrices as

\[ H(z_1, z_2) = c_1(z_1 I_{N_1} - A_1)^{-1} b_1 + c_2(z_2 I_{N_2} - A_4)^{-1} b_2 + c_2(z_2 I_{N_2} - A_4)^{-1} A_3(z_1 I_{N_1} - A_1)^{-1} b_1 + d. \] (2)

The transfer function \( H(z_1, z_2) \) is invariant under similarity transformation of the state. Let \( T_1 \) and \( T_4 \) be \( N_1 \times N_1 \) and \( N_2 \times N_2 \) nonsingular matrices, and \( T = \text{diag}(T_1, T_4) \). Consider the transformation of the state vector as

\[
\begin{bmatrix}
\mathbf{x}^h(n_1, n_2) \\
\mathbf{x}^v(n_1, n_2)
\end{bmatrix} = \begin{bmatrix}
T_1^{-1} x^h(n_1, n_2) \\
T_4^{-1} x^v(n_1, n_2)
\end{bmatrix}.
\] (3)

Then, a 2-D digital filter with the new states \( \mathbf{x}^h(n_1, n_2) \) and \( \mathbf{x}^v(n_1, n_2) \) can be expressed by \( [T_1^{-1} A_1 T_1, T_4^{-1} A_3 T_4, T_4^{-1} A_3 T_4, T_4^{-1} b_1, T_4^{-1} b_2, c_1 T_1, c_2 T_4, d] \) and its transfer function is equal to Eq. (2).

### 2.2. Controllability and observability grammians and second-order modes

The horizontal controllability and observability grammians \( K^h \) and \( W^h \) and the vertical controllability and observability grammians \( K^v \) and \( W^v \) of the 2-D stable digital filter of Eq. (1) are the solutions of the following Lyapunov equations, respectively [2]:

\[
\begin{align*}
K^h &= A_1 K^h A_1^T + b_1 b_1^T \\
K^v &= A_4 K^v A_4^T + A_3 K^h A_3^T + b_2 b_2^T \\
W^h &= A_1^T W^h A_1 + A_3^T W^v A_3 + c_1^T c_1 \\
W^v &= A_4^T W^v A_4 + c_2^T c_2.
\end{align*}
\] (4-7)

\( K^h \) and \( W^h \) are the controllability and observability grammians for the horizontal state vector \( x^h \), respectively, and \( K^v \) and \( W^v \) are the controllability and observability grammians for the vertical state vector \( x^v \). The controllability and observability grammians are symmetric and positive definite if the filter is stable. They play very important roles in analysis of roundoff noise and coefficient sensitivity in 2-D digital filters. They are also called the covariance and noise matrices respectively in quantization error analysis.

Let \( (\theta_1^h)^2 \) and \( (\theta_1^v)^2 \) be the eigenvalues of the products \( K^h W^h \) and \( K^v W^v \), respectively. Then, the square roots \( \theta_1^h \) and \( \theta_1^v \) are the eigenvalues of the horizontal and vertical second-order modes of the 2-D digital filter, respectively.

The second-order modes of a 2-D digital filter are invariant under similarity transformation. This is shown as the following: Let the new grammians of a transformed 2-D digital filter by \( T = \text{diag}(T_1, T_4) \) be \( \mathbf{K}^h, \mathbf{W}^h, \mathbf{K}^v, \) and \( \mathbf{W}^v \). Then, we know \( \mathbf{K}^h \mathbf{W}^h = T_1^{-1} (K^h W^h) T_1 \) and \( \mathbf{K}^v \mathbf{W}^v = T_4^{-1} (K^v W^v) T_4 \), which show that the eigenvalues of \( \mathbf{K}^h \mathbf{W}^h \) are equal to those of \( K^h W^h \), and the eigenvalues of \( \mathbf{K}^v \mathbf{W}^v \) are equal to those of \( K^v W^v \). Thus, the second-order modes are invariant under similarity transformation.

### 2.3. Frequency transformations

Let \( H(z_1, z_2) \) be the prototype transfer function of separable denominator, which is generally lowpass. The frequency transformation in 2-D case is given by [5]

\[
H_d(z_1, z_2) = \frac{H(z_1, z_2)}{\left| z_1 - F_1(z_1), z_2 - F_2(z_2) \right|}, \quad i = 1, 2
\] (8)

\[
F_i(z_i) = \pm \prod_{k=1}^{M_i} \left( \frac{z_i - \xi_{ik}}{1 - \xi_{ik} z_i} \right), \quad i = 1, 2
\] (9)

where \( H_d(z_1, z_2) \) is a 2-D digital filter obtained by the frequency transformations \( F_1(z_1) \) and \( F_2(z_2) \), where \( 1/F_1(z_1) \) and \( 1/F_2(z_2) \) are \( M_1 \)-th order and \( M_2 \)-th order \( 1 \)-D allpass filters and \( \xi_{ik} \) are possibly complex with all \( |\xi_{ik}| < 1 \) for stability and “=” denotes complex conjugate. From Eqs. (8) and (9), the 2-D frequency transformation can be applied horizontally and vertically to the prototype 2-D digital filter of separable denominator, and the resultant \( H_d(z_1, z_2) \) is also of separable denominator, but its order is \( M_1 \times N_1 \) for horizontal direction and \( M_2 \times N_2 \) for vertical direction.

### 3. TRANSFORMED 2-D DIGITAL FILTERS AND GRAMMIANS

#### 3.1. Expression of frequency transformation in terms of state-space equations

We next consider the state-space equations for the transformed 2-D digital filter \( H_d(z_1, z_2) \) by the 2-D frequency transformation \( F_1(z_1) \) and \( F_2(z_2) \), that is, we obtain the state-space equations (\( \mathbf{A}_1, \mathbf{A}_3, \mathbf{A}_4, \mathbf{B}_1, \mathbf{B}_2, \mathbf{C}_1, \mathbf{C}_2, \mathbf{D} \)) satisfying

\[
H_d(z_1, z_2) = H(F_1(z_1), F_2(z_2))
\]

\[
= c_1(z_1 I_{N_1 M_1} - \mathbf{A}_1)^{-1} \mathbf{B}_1 + c_2(z_2 I_{N_2 M_2} - \mathbf{A}_4)^{-1} \mathbf{B}_2 + c_2(z_2 I_{N_2 M_2} - \mathbf{A}_4)^{-1} \mathbf{A}_3(z_1 I_{N_1 M_1} - \mathbf{A}_1)^{-1} \mathbf{B}_1 + \mathbf{D}.
\] (10)

To this end, we first let \((\alpha_1, \beta_1, \gamma_1, \delta_1)\) be a 1-D digital filter for \( 1/F_1(z_1) \) and \((\alpha_2, \beta_2, \gamma_2, \delta_2)\) be a 1-D digital filter for \( 1/F_2(z_2) \), that is,

\[
1/F_i(z_i) = \gamma_i(z_i I_{M_i} - \alpha_i)^{-1} \beta_i + \delta_i, \quad i = 1, 2
\] (11)

As shown in Fig. 2, the 2-D frequency transformation can be considered as substitution of the allpass filters \( 1/F_1(z_1) \) and \( 1/F_2(z_2) \) into the delays of \( z_1^{-1} \) and \( z_2^{-1} \), respectively. Thus, the transformed filter \( (\mathbf{A}_1, \mathbf{A}_3, \mathbf{A}_4, \mathbf{B}_1, \mathbf{B}_2, \mathbf{C}_1, \mathbf{C}_2, \mathbf{D}) \) can be represented by the original 2-D digital filter \( (\mathbf{A}_1, \mathbf{A}_3, \mathbf{A}_4, \mathbf{B}_1, \mathbf{B}_2, \mathbf{C}_1, \mathbf{C}_2, \mathbf{D}) \) and 1-D digital filters \((\alpha_1, \beta_1, \gamma_1, \delta_1)\) and \((\alpha_2, \beta_2, \gamma_2, \delta_2)\).
3.2. Grammians of the transformed 2-D digital filter

Let the controllability and observability grammians of the transformed 2-D filter by frequency transformations be $\mathcal{K}^h$, $\mathcal{K}'$, $\mathcal{W}^h$ and $\mathcal{W}'$. They are the solutions of the following Lyapunov equations:

\begin{align*}
\mathcal{K}^h & = \alpha_i \mathcal{K}_F \alpha_i^t + \beta_i \beta_i^t \quad (i = 1, 2) \\
\mathcal{K}' & = \mathcal{K}_F' \quad \text{(25)}
\end{align*}

On the other hand, $\mathcal{K}'$ and $\mathcal{W}^h$ are obtained as

\begin{align*}
\mathcal{K}' & = \mathcal{K}' \otimes \mathcal{K}'_F', \quad \mathcal{W}^h = \mathcal{W}^h \otimes \mathcal{K}^{-1}_F \\
\mathcal{K}^h & = A_1 \mathcal{K}' A_1^t + A_2 \mathcal{K}' A_2^t + b_2 b_2^t + V \\
\mathcal{W}^h & = A_1 \mathcal{W}^h A_1 + A_2 \mathcal{W}^h A_2 + c_1 c_1 + H.
\end{align*}

where $\mathcal{K}_F$'s are the controllability grammians of the all-pass filters $1/F_i(z)$. They are given by the solutions of the following Lyapunov equations:

\begin{align*}
\mathcal{K}^h & = \alpha_i \mathcal{K}_F \alpha_i^t + \beta_i \beta_i^t \quad (i = 1, 2) \\
\mathcal{K}' & = \mathcal{K}_F' \quad \text{(25)}
\end{align*}

The matrices $H$ and $V$ are set to be

\begin{align*}
H & = \delta_2 \left[ R \left( A_1 \mathcal{W}^h A_1 + 2 \delta_2 A_1^t \mathcal{W}^h A_1 \right)\right] R \\
V & = \delta_1 \left[ P \mathcal{K}_F A_2^t + \mathcal{K}_F A_2^t P \right] + P b_2 b_2^t + b_2 b_2^t P^t \\
& \text{(29)}
\end{align*}

where we let $R = (I - \delta_2 A_2^t)^{-1} A_2^t$ and $P = A_2 (I - \delta_2 A_2)^{-1} I$.

4. INVARIANCE OF THE SECOND-ORDER MODES

4.1. Invariant Conditions

From Eqs. (24) and (26), we have

\begin{align*}
\mathcal{K}^h \mathcal{W}^h &= (\mathcal{K}^h \mathcal{W}^h) \otimes I_{M_1} \\
\mathcal{K}' \mathcal{W}' &= (\mathcal{K}' \mathcal{W}') \otimes I_{M_2} \\
\end{align*}

where $I_{M_1}$ and $I_{M_2}$ are the identity matrices with size of $M_1 \times M_1$ and $M_2 \times M_2$, respectively. Eq. (31) shows that the eigenvalues of $\mathcal{K}^h \mathcal{W}^h$ of the transformed 2-D filter are $M_1$ copies of the eigenvalues of $\mathcal{K}^h \mathcal{W}^h$.

Therefore, if $\mathcal{W}^h = \mathcal{W}^h$, then the eigenvalues of $\mathcal{K}^h \mathcal{W}^h$ are equal to the eigenvalues of $\mathcal{K}^h \mathcal{W}^h$ with duplication. Comparing Eq. (22) for $\mathcal{W}^h$ with Eq. (28) for $\mathcal{W}^h$, we observe that the condition $\tilde{\mathcal{W}}^h = \mathcal{W}^h$ is equivalent to the condition $H = 0$, which holds for $\delta_2 = 0$ in Eq. (29). Thus, the horizontal second-order modes, which are square roots of the eigenvalues of $\mathcal{K}^h \mathcal{W}^h$, are invariant under frequency transformations by the allpass filter $1/F_2(z_2)$ having $\delta_2 = 0$.

In Eq. (11), $\delta_2$ is the direct feedthrough scalar of the 1-D state-space equations of $1/F_2(z_2)$. The transfer function of which direct feedthrough scalar $\delta_2 = 0$ is called strictly
proper. Thus, the horizontal second-order modes are invariant under frequency transformations by any strictly proper vertical allpass filter. Similarly, the eigenvalues of \( \mathbf{K}' \odot \mathbf{W}^v \) are \( M_2 \) copies of the eigenvalues of \( \mathbf{K} \odot \mathbf{W}^v \), the eigenvalues of \( \mathbf{K}' \odot \mathbf{W}^v \) are equal to the eigenvalues of \( \mathbf{K} \odot \mathbf{W}^v \) with duplication, if \( \mathbf{K} = \mathbf{K}' \). This condition is equal to the condition \( \mathbf{V} = \mathbf{0} \) which holds for \( \delta_1 = 0 \). Thus, the vertical second-order modes are invariant under frequency transformations with any strictly proper horizontal allpass filter.

4.2. Invariance of second-order modes in some practical cases

We next give some practical frequency transformations under which the second-order modes are invariant.

4.2.1. Lowpass-Bandpass and Lowpass-Bandstop transformations

The transfer function of a lowpass-bandpass transformation is given by

\[
\frac{1}{F_1(z)} = \frac{z_i^{-1} - \frac{2\xi_i k_i}{k_i + 1} z_i^{-1} + \frac{k_i - 1}{k_i + 1}}{z_i^{-1} - \frac{2\xi_i k_i}{k_i + 1} z_i^{-1} + 1}, \quad i = 1, 2
\]  

(32)

where \( \xi_i \) and \( k_i \) are parameters determining the center frequency and bandwidth of the transformed bandpass filter. From the above equation, we have the direct feedthrough scalar \( \delta_i \) as \( \delta_i = -\frac{k_i - 1}{k_i + 1}, \quad i = 1, 2 \). In the case of \( \delta_i = 0 \), which means \( k_i = 1 \), we have the following frequency transformation:

\[
\frac{1}{F_1(z)} = -z_i^{-1} - \frac{z_i^{-1} - \xi_i}{-\xi_i z_i^{-1} + 1}, \quad i = 1, 2
\]  

(33)

which are known as the transfer function of lowpass-bandpass frequency transformation keeping the same bandwidth as that of the prototype lowpass filter. Thus, the second-order modes are invariant under the lowpass-bandpass transformation which provides the same bandwidth as that of the prototype lowpass filter. The same discussion can be made on lowpass-bandstop transformations.

4.2.2. Lowpass-Lowpass and Lowpass-Highpass transformations

Consider the following transfer function of vertical lowpass-lowpass transformation:

\[
\frac{1}{F_2(z_2)} = \frac{z_2^{-1} - \xi_2}{1 - \xi_2 z_2^{-1}} = \frac{(1 - \xi_2^2) z_2^{-1}}{1 - \xi_2 z_2^{-1}} - \xi_2.
\]

(34)

The state-space expression for the above transfer function has the direct feedthrough scalar \( \delta_2 = -\xi_2 \). Thus, if \( \delta_2 = -\xi_2 = 0 \), then the transfer function of lowpass-lowpass vertical frequency transformation is strictly proper, and the horizontal second-order modes are invariant under the frequency transformation. But, \( \delta_2 = -\xi_2 = 0 \) in Eq. (34) reduces the transfer function \( 1/F_2(z_2) \) into \( z_2^{-1} \), which is clearly an identity frequency transformation and has no effect on the vertical frequency characteristic.

Therefore, the horizontal second-order modes are invariant when only horizontal transformation is applied. The role of horizontal and vertical directions is changeable. The similar discussion can be made on lowpass-highpass transformation.

5. CONCLUDING REMARKS

This paper has discussed the invariance of the second-order modes of 2-D separable denominator digital filters. The second-order modes are invariant if the allpass transfer function for the frequency transformation is strictly proper. Our numerical examples, which are not shown in this paper for limited space, confirm the invariance.

In the 2-D non-separable denominator case, which interests many researchers in M-D digital filters, we have already confirmed the same property of the invariance of the second-order modes by numerical experiments. But, it remains to be a future work to give a theoretical proof for the invariance.

6. REFERENCES


