Synthesis of High Accuracy Digital Filter Structures Based on State-Space Representations

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Abstract: This paper introduces synthesis of high accuracy digital filter structures based on the state-space representations. We mainly focus on synthesis of the minimum $L_2$-sensitivity realizations. We propose closed form expressions of the minimum $L_2$-sensitivity realizations without $L_2$-scaling constraints and subject to $L_2$-scaling constraints, for second-order state-space digital filters. Furthermore, we show the absence of limit cycles of the minimum $L_2$-sensitivity realizations theoretically and numerically.

Keywords— State-Space Digital Filters, $L_2$-Sensitivity, $L_2$-Scaling Constraints, Limit Cycles

1. Introduction

On the fixed-point implementation of digital filters, undesirable finite-word-length (FWL) effects arise due to the coefficient truncation and arithmetic roundoff. These FWL effects must be reduced as small as possible because such effects may cause serious degradation of characteristic of digital filters. In order to reduce these undesirable FWL effects, it is important to synthesize high accuracy digital filter structures which have low-sensitivity, low-roundoff noise and no limit cycles.

$L_2$-sensitivity is one of the evaluation functions which evaluate the coefficient quantization effects of state-space digital filters [1, 2]. To the $L_2$-sensitivity minimization problem, Yan et al. [1] and Hinamoto et al. [2] proposed solutions using iterative calculations. Both of the solutions in [1] and [2] try to solve nonlinear equations by successive approximation. Their solutions do not guarantee that the $L_2$-sensitivity surely converges to the minimum $L_2$-sensitivity since their solutions are not analytical solutions. It is necessary to derive some analytical solutions to the $L_2$-sensitivity minimization problem in order to guarantee that their conventional solutions surely derive the minimum $L_2$-sensitivity.

In this paper, we introduce our analytical approaches for synthesis of the minimum $L_2$-sensitivity realizations for second-order digital filters [3–6]. Second-order digital filters play important role in implementation of higher-order digital filters as basic sections or sub-filters. Thus, second-order digital filters are useful and important realizations in considering the $L_2$-sensitivity minimization. The proposed closed form approaches can guarantee that the $L_2$-sensitivity obtained by the iterative algorithm of the conventional method surely converges to the optimal solution. Furthermore, we have shown the absence of limit cycles in the minimum $L_2$-sensitivity realization [7, 8].

2. State-Space Digital Filters and $L_2$-Sensitivity

2.1 State-Space Digital Filters

For a given $N$th-order transfer function $H(z)$, a state-space digital filter can be described by the following state-space equations:

\begin{align}
\dot{x}(n+1) &= A x(n) + b u(n) \\
y(n) &= c x(n) + d u(n)
\end{align}

where $x(n)$ is a state variable vector, $u(n)$ is a scalar input, $y(n)$ is a scalar output, and $(A, b, c, d)$ are real constant matrices called coefficient matrices. The block diagram of the state-space digital filter $(A, b, c, d)$ is shown in Fig. 1. The transfer function $H(z)$ of the state-space digital filter $(A, b, c, d)$ is defined by $H(z) = c(zI - A)^{-1}b + d$.

2.2 $L_2$-Sensitivity

The $L_2$-sensitivity of the filter $H(z)$ with respect to the realization $(A, b, c, d)$ is defined by

\begin{equation}
S(A, b, c) = \left\| \frac{\partial H(z)}{\partial A} \right\|^2_2 + \left\| \frac{\partial H(z)}{\partial b} \right\|^2_2 + \left\| \frac{\partial H(z)}{\partial c} \right\|^2_2 = \text{tr}(W_0)\text{tr}(K_0) + \text{tr}(W_0) + \text{tr}(K_0) + 2\sum_{i=1}^{\infty} \text{tr}(W_i)\text{tr}(K_i)
\end{equation}

where $\| \cdot \|_2$ denotes the $L_2$-norm. The general controllability Gramian $K_i$ and the general observability Gramian $W_i$ in (3) are defined by solutions of the following Lyapunov equations:

\begin{align}
K_i &= A K_i A^T + \frac{1}{2} \left( A^T b b^T + b b^T (A^T)^i \right) \\
W_i &= A^T W_i A + \frac{1}{2} \left( c^T c A^i + (A^T)^i c^T c \right)
\end{align}

Figure 1. Block diagram of a state-space digital filter.
for \( i = 0, 1, 2, \cdots \), respectively. The general controllability and observability Gramians are natural expansions of the controllability and observability Gramians, respectively. Letting \( i = 0 \) in Eqs. (4) and (5), we have the Lyapunov equations to obtain the controllability Gramian \( K_0 \) and the observability Gramian \( W_0 \) as follows:

\[
K_0 = AK_0A^T + bb^T \tag{6}
\]

\[
W_0 = A^TW_0A + c^Te. \tag{7}
\]

The value of \( L_2 \)-sensitivity depends on not only the transfer function \( H(z) \) but also the coordinate transformation matrix \( T \). Under the coordinate transformation defined by \( x(n) = T^{-1}x(n) \), we have the \( L_2 \)-sensitivity of the transformed filter structure \((T^{-1}AT, T^{-1}b, cT, d)\) as follows:

\[
S(P) = \text{tr}(W_0P) + \text{tr}(K_0P^{-1}) + 2\sum_{i=1}^{\infty} \text{tr}(W_iP)\text{tr}(K_iP^{-1}) \tag{8}
\]

where \( P \) is a positive definite symmetric matrix defined by \( P = TT^T \) [2].

3. Synthesis of the Minimum \( L_2 \)-Sensitivity Realizations of State-Space Digital Filters

This section reviews closed form solutions to the \( L_2 \)-sensitivity minimization of second-order state-space digital filters without \( L_2 \)-scaling constraints [3,4] and subject to \( L_2 \)-scaling constraints [5,6].

3.1 Closed Form Expressions of Balanced Realizations of Second-Order Digital Filters

First, our groups have proposed closed form expressions of the balanced realizations of second-order digital filters [9,10]. These results would be necessary in synthesis of the minimum \( L_2 \)-sensitivity realizations. The balanced realization \((A_b, b_b, c_b, d_b)\) is a filter structure of which controllability and observability Gramians are equal as follows:

\[
K_0^{(b)} = W_0^{(b)} = \Theta, \quad \Theta = \text{diag}(\theta_1, \cdots, \theta_N) \tag{9}
\]

where the parameters \( \theta_i(i = 1, \cdots, N) \) are the second order modes of the filter \( H(z) \). All of second-order digital filters with real filter coefficients can be categorized into the following three cases:

- **Case 1**: Poles are complex conjugate.
- **Case 2**: Poles are real and distinct. \( \tag{10} \)
- **Case 3**: Poles are real and multiple.

For these three cases, we have derived closed form expressions of the balanced realizations [9,10].

3.2 The Minimum \( L_2 \)-Sensitivity Realization without \( L_2 \)-Scaling Constraints

We have proposed closed form expressions of the minimum \( L_2 \)-sensitivity realizations without \( L_2 \)-scaling constraints [3,4]. We adopt the balanced realization \((A_b, b_b, c_b, d_b)\) as the initial realization to synthesize the minimum \( L_2 \)-sensitivity realization. Under this condition, the \( L_2 \)-sensitivity \( S(P) \) in Eq. (8) is rewritten as

\[
S(P) = \text{tr}(W_0^{(b)}P)\text{tr}(K_0^{(b)}P^{-1}) + \text{tr}(W_0^{(b)}P) + \text{tr}(K_0^{(b)}P^{-1}) + 2\sum_{i=1}^{\infty} \text{tr}(W_i^{(b)}P)\text{tr}(K_i^{(b)}P^{-1}) \tag{11}
\]

and thus, the \( L_2 \)-sensitivity minimization problem without \( L_2 \)-scaling constraints is formulated as [2]

\[
\min_P S(P) \quad \text{in Eq. (11)} \tag{12}
\]

where \( P \) is an arbitrary positive definite symmetric matrix.

We have newly proved that the positive definite symmetric matrix \( P \) which gives the global minimum of the \( L_2 \)-sensitivity \( S(P) \) is expressed as

\[
P = \begin{bmatrix} \cosh(p) & \sinh(p) \\ \sinh(p) & \cosh(p) \end{bmatrix} \tag{13}
\]

where \( p \) is a real scalar variable. Substituting Eq. (13) into Eq. (11) gives the closed form expression of the \( L_2 \)-sensitivity \( S(p) \) as

\[
S(P) = S(p) = \sum_{n=-2}^{2} s_ne^{np}. \tag{14}
\]

These coefficients \( s_n \)'s are computed directly from the transfer function \( H(z) \). The parameter \( p \) which minimizes \( S(p) \) in Eq. (14) can be derived by solving the following equation with respect to \( p \):

\[
\frac{\partial S(p)}{\partial p} = \sum_{n=-2}^{2} ns_ne^{np} = 0. \tag{15}
\]

Eq. (15) can be solved analytically since it is a fourth-degree polynomial equation of constant coefficients. Eq. (15) has four solutions, from which the real solution \( p_{\text{opt}} \) is adopted to derive the optimal positive definite symmetric matrix \( P_{\text{opt}} \) as

\[
P_{\text{opt}} = \begin{bmatrix} \cosh(p_{\text{opt}}) & \sinh(p_{\text{opt}}) \\ \sinh(p_{\text{opt}}) & \cosh(p_{\text{opt}}) \end{bmatrix}. \tag{16}
\]

Once the optimal positive definite symmetric matrix \( P_{\text{opt}} \) is derived, the optimal coordinate transformation matrix \( T_{\text{opt}} \) is calculated as

\[
T_{\text{opt}} = P_{\text{opt}}^{1/2}U \tag{17}
\]

where \( U \) is an arbitrary orthogonal matrix. Finally, the minimum \( L_2 \)-sensitivity realization without \( L_2 \)-scaling constraints \((A_{\text{opt}}, b_{\text{opt}}, c_{\text{opt}}, d_{\text{opt}})\) is synthesized as

\[
\begin{bmatrix} A_{\text{opt}} \\ b_{\text{opt}} \\ c_{\text{opt}} \\ d_{\text{opt}} \end{bmatrix} = \begin{bmatrix} T_{\text{opt}}^{-1}A \end{bmatrix}_{\text{opt}} T_{\text{opt}}^{-1}b_{\text{opt}} \begin{bmatrix} c_{\text{opt}} \\ T_{\text{opt}}b \end{bmatrix} d_{\text{opt}}. \tag{18}
\]
3.3 The Minimum $L_2$-Sensitivity Realizations Subject to $L_2$-Scaling Constraints

We have proposed closed form expressions of the minimum $L_2$-sensitivity realizations subject to $L_2$-scaling constraints [5, 6]. In order to prevent the overflow of state variables, the variance of state variables must be unity under the white Gaussian input with zero mean and unit variance as shown in Fig. 2. We adopt the input normal realization $(A_i, b_i, c_i, d_i)$ given by

$$(A_i, b_i, c_i, d_i) = (\Theta^{-\frac{1}{2}} A_i \Theta^\frac{1}{2}, \Theta^{-\frac{1}{2}} b_i, c_i \Theta^\frac{1}{2}, d_i)$$  

(19)

as the initial realization. Under this condition, the $L_2$-sensitivity $S(P)$ in Eq. (8) is rewritten as

$$S(P) = \text{tr}(W_0^{(i)} P + (K_0^{(i)} P)^{-1}) + \text{tr}(W_0^{(i)} P + (K_0^{(i)} P)^{-1}) + 2 \sum_{i=1}^{\infty} \text{tr}(W_i^{(i)} P + (K_i^{(i)} P)^{-1})$$  

(20)

and thus, the $L_2$-sensitivity minimization problem subject to $L_2$-scaling constraints is formulated as [11]

$$\min_P S(P) \text{ in Eq. (20) subject to } \text{tr}(K_0^{(i)} P^{-1}) = \text{tr}(P^{-1}) = N.$$  

(21)

In order to reduce the constraint condition in the problem (21), we newly propose a variable transformation defined as

$$P = \frac{\text{tr}(Q^{-1})}{N} Q$$  

(22)

where $Q$ is a newly introduced positive definite symmetric matrix. Substituting Eq. (22) into Eq. (20), we have a novel expression of $L_2$-sensitivity $\tilde{S}(Q)$ with variable matrix $Q$ given by

$$\tilde{S}(Q) = N + \frac{N+1}{N} \text{tr}(W_0^{(i)} Q + (K_0^{(i)} Q)^{-1}) + 2 \sum_{i=1}^{\infty} \text{tr}(W_i^{(i)} Q + (K_i^{(i)} Q)^{-1}).$$  

(23)

Under the variable transformation in Eq. (22), the constrained optimization problem (21) is reduced to the unconstrained optimization problem as

$$\min_Q \tilde{S}(Q) \text{ in Eq. (23)}$$  

where $Q$ is an arbitrary positive definite symmetric matrix.

We newly proved that the positive definite symmetric matrix $Q$ which gives the global minimum of the $L_2$-sensitivity $\tilde{S}(Q)$ is expressed as

$$Q = \Theta^{-\frac{1}{2}} \begin{bmatrix} \cosh(q) & \sinh(q) \\ \sinh(q) & \cosh(q) \end{bmatrix} \Theta^{-\frac{1}{2}}$$  

(25)

where $q$ is a real scalar variable. Substituting Eq. (25) into Eq. (23) gives the closed form expression of the $L_2$-sensitivity $\tilde{S}(q)$ as

$$\tilde{S}(Q) = \tilde{S}(q) = \sum_{n=-1}^{1} \tilde{s}_n e^{2 n q}.$$  

(26)

These coefficients $\tilde{s}_n$'s are computed directly from the transfer function $H(z)$. The parameter $q_{\text{opt}}$ which minimizes $\tilde{S}(q)$ in Eq. (26) is derived as

$$q_{\text{opt}} = \frac{1}{4} \log \left( \frac{\tilde{s}_1}{\tilde{s}_0} \right).$$  

(27)

The coordinate transformation matrix $T_{\text{opt}}$, which gives the minimum $L_2$-sensitivity realization, is derived as follows:

$$\mathcal{Q}_{\text{opt}} = \Theta^{-\frac{1}{2}} \begin{bmatrix} \cosh(q_{\text{opt}}) & \sinh(q_{\text{opt}}) \\ \sinh(q_{\text{opt}}) & \cosh(q_{\text{opt}}) \end{bmatrix} \Theta^{-\frac{1}{2}}$$  

(28)

$$P_{\text{opt}} = \frac{\text{tr}(Q_{\text{opt}}^{-1})}{2} \mathcal{Q}_{\text{opt}}$$  

(29)

$$T_{\text{opt}} = \mathcal{P}_{\text{opt}}^{\frac{1}{2}} U.$$  

(30)

Finally, the minimum $L_2$-sensitivity realization subject to $L_2$-scaling constraints $(\mathcal{A}_{\text{opt}}, \mathcal{B}_{\text{opt}}, \mathcal{C}_{\text{opt}}, \mathcal{D}_{\text{opt}})$ is given by

$$\begin{bmatrix} \mathcal{A}_{\text{opt}} \\ \mathcal{B}_{\text{opt}} \\ \mathcal{C}_{\text{opt}} \\ \mathcal{D}_{\text{opt}} \end{bmatrix} = \begin{bmatrix} T_{\text{opt}}^{-1} A_T T_{\text{opt}} & T_{\text{opt}}^{-1} b_T \\ c_T T_{\text{opt}} & d_T \end{bmatrix}.$$  

(31)

4. On the Absence of Limit Cycles of the Minimum $L_2$-Sensitivity Realizations

We have shown the absence of limit cycles of the minimum $L_2$-sensitivity realizations. For the minimum $L_2$-sensitivity realizations without $L_2$-scaling constraints, we have given a theoretical proof of the absence of limit cycles [7]. For the minimum $L_2$-sensitivity realizations subject to $L_2$-scaling constraints, we have given a numerical conjecture of the absence of limit cycles [8].

4.1 The Minimum $L_2$-Sensitivity Realizations without $L_2$-Scaling Constraints

In chapter 3, we solve the $L_2$-sensitivity minimization problem in (12) adopting the balanced realization
where we have given proof of the absence of limit cycles. However, in Ref. [8],

\[ T_{opt} = P_{opt}^2 U = R^T B_{opt} RU \]  

(33)

where \( R \) is an orthogonal matrix, and \( B_{opt} \) is a positive definite diagonal matrix. In the above expression, \( U \) is an arbitrary orthogonal matrix. We show that the minimum \( L_2 \)-sensitivity realization without \( L_2 \)-scaling constraints does not generate limit cycles if we specify the orthogonal matrix as \( U = R^T \), which yields

\[ \tilde{T}_{opt} = R^T B_{opt}^2. \]  

(34)

In other words, the minimum \( L_2 \)-sensitivity realization \((\tilde{A}_{opt}, \tilde{b}_{opt}, \tilde{c}_{opt}, \tilde{d}_{opt})\), obtained by the coordinate transformation by \( T_{opt} \) such as

\[ (\tilde{A}_{opt}, \tilde{b}_{opt}, \tilde{c}_{opt}, \tilde{d}_{opt}) \]

\[ = (T_{opt}^{-1} A_b T_{opt}, T_{opt}^{-1} b_b, c_b T_{opt}, d_b) \]  

(35)

does not generate limit cycles. The controllability Gramian \( \tilde{K}_0^{(opt)} \) and the observability Gramians \( \tilde{W}_0^{(opt)} \) of the minimum \( L_2 \)-sensitivity realization \((\tilde{A}_{opt}, \tilde{b}_{opt}, \tilde{c}_{opt}, \tilde{d}_{opt})\) satisfy

\[ \tilde{W}_0^{(opt)} = B_{opt} \tilde{K}_0^{(opt)} B_{opt} \]  

(36)

which is a sufficient condition for the absence of limit cycles [12].

4.2 The Minimum \( L_2 \)-Sensitivity Realizations Subject to \( L_2 \)-Scaling Constraints

For the minimum \( L_2 \)-sensitivity realizations subject to \( L_2 \)-scaling constraints, we have not accomplished theoretical proof of the absence of limit cycles. However, in Ref. [8], we have given conjecture on the absence of limit cycles of second-order digital filters with minimum \( L_2 \)-sensitivity subject to \( L_2 \)-scaling constraints by numerical experiments. Giving a theoretical proof for these numerical experiments is our future work.

5. Conclusions

In this paper, we introduce our analytical approaches for synthesis of the minimum \( L_2 \)-sensitivity realizations for second-order digital filters. The proposed closed form approaches can guarantee that the \( L_2 \)-sensitivity obtained by the iterative algorithm of the conventional method surely converges to the optimal solution. Furthermore, we have shown the absence of limit cycles in the minimum \( L_2 \)-sensitivity realization.

References


