Analysis of Frequency Estimation MSE for All-pass-Based Adaptive IIR Notch Filters With Normalized Lattice Structure

Shunsuke Koshita, Yuki Noguchi, Masahide Abe, Masayuki Kawamata
Department of Electronic Engineering, Graduate School of Engineering, Tohoku University, Japan

Abstract

This paper theoretically analyzes the Mean Square Error (MSE) on the steady-state frequency estimation realized by the all-pass-based adaptive notch filtering algorithms with the normalized lattice structure. The adaptive algorithms to be considered are the Simplified Lattice Algorithm (SLA) proposed by Regalia and the Affine Combination Lattice Algorithm (ACLA) proposed by the authors. For these two algorithms, we derive the frequency estimation MSE in closed-form. The derivation is based on construction of a linear time-invariant model for generation of frequency estimation error, and division of this model into two submodels of which output signals are statistically uncorrelated to each other. This strategy leads to more accurate theoretical MSE expressions than the direct use of the existing analysis methods. Simulation results demonstrate that our theoretical MSE expressions are in very good agreement with the simulated MSE values.

Keywords: adaptive notch filter, all-pass filter, normalized lattice structure, mean square error, simplified lattice algorithm, Affine combination lattice algorithm

1. Introduction

Adaptive notch filters [1] are the time-variant notch filters of which frequency characteristics, e.g. notch frequency, are controlled by adaptive al-
gorithms. Such adaptive algorithms are capable of detecting the unknown frequency of a sinusoid that is immersed in white noise. This mechanism can automatically remove or enhance the unknown sinusoid, leading to many practical applications such as radar, sonar, removal of narrowband interference in communication systems, and howling suppressor in speech processing systems. There exist many techniques for design of adaptive notch filters. Among them, the adaptive notch filters using second-order IIR transfer functions have attracted a lot of researchers because the IIR transfer functions can attain desired frequency characteristics at smaller computational complexity, compared with the FIR counterparts.

For the transfer functions of second-order adaptive IIR notch filters, there exist many design methods. Some of the well-known design methods are the IIR notch filters with constrained poles and zeros [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13], the bilinear IIR notch filters [14, 15, 16, 17], and the IIR notch filters based on all-pass filter [1, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28]. In this paper we consider the IIR notch filters based on all-pass filter. Specifically, we focus on the all-pass-based IIR adaptive notch filters with the normalized lattice structure [1, 19, 28] because they have many attractive properties in practice. For example, the all-pass-based adaptive notch filtering achieves unbiased frequency estimation [1, 19]. Moreover, the lattice structures are known to be of much better performance with respect to the finite wordlength effects than the direct form and, in addition, the normalized lattice structure shows the best performance in the family of the lattice structures [29]. Furthermore, for the all-pass-based notch filter with the normalized lattice structure, two useful adaptive algorithms are available to achieve much faster convergence speed than the standard gradient decent algorithm, especially in the case where the frequency of the sinusoid is far from the notch frequency. One of these algorithms is the Simplified Lattice Algorithm (SLA) [1, 19], which is based on a very simple update equation using the state variable of the normalized lattice structure. The other is the Affine Combination Lattice Algorithm (ACLA) [28], which further accelerates the convergence speed of the SLA by using the affine combination of the gradient decent algorithm and the SLA. In view of the aforementioned advantages, the all-pass-based adaptive notch filters with the normalized lattice structure are considered as a promising technique in signal processing.

The main objective of this paper is to theoretically analyze the Mean Square Error (MSE) on the steady-state frequency estimation introduced by the SLA and the ACLA, and to derive the frequency estimation MSE in
closed-form expression. Analysis of the MSE plays an important role in performance evaluation of adaptive notch filters because the theoretical value of the frequency estimation MSE tells us the accuracy of the frequency estimation in adaptive notch filtering. In addition, if a closed-form expression of the frequency estimation MSE is obtained, it enables us to determine optimal parameters of the adaptive notch filtering that achieve the best convergence speed and the best estimation accuracy under a given error tolerance. Although many results on the MSE analysis are presented in the literature [4, 5, 6, 8, 9, 10, 12, 16, 21, 22, 24], nothing has been reported on this topic for the SLA and the ACLA: to the best of the authors’ knowledge, MSE analysis in the all-pass-based adaptive notch filters with the normalized lattice structure is considered for only the gradient decent algorithm [24]. Therefore, it is necessary to investigate the theoretical MSE for the SLA and the ACLA.

It should be noted that derivation of the theoretical MSE in closed-form is not usually a straightforward task. As was pointed out in [4], the analysis method for a certain adaptive algorithm cannot be directly applied to other adaptive algorithms. This is also true in the cases of the SLA and the ACLA: as will be demonstrated in Section 5, the theoretical MSEs given by the direct use of the existing analysis methods significantly deviate from the simulated MSE values. Therefore, derivation of high-accuracy MSE expressions for the SLA and the ACLA requires a new strategy as well as the use of the existing methods. To this end, we shall derive the MSE by means of combination of some of the conventional analysis methods and our new strategy. In our method, we first construct an error generation model with the help of [16, 22]. Then we carry out a new strategy: we divide this model into two submodels of which output signals are uncorrelated to each other. To these submodels we apply the analysis methods of [4, 16, 24] and obtain the MSE expressions for the SLA and the ACLA. These procedures lead to more accurate MSE expressions than the direct use of the existing analysis methods.

The rest of this paper is organized as follows. Section 2 reviews the fundamentals of adaptive notch filtering and introduces the SLA and the ACLA. Sections 3 and 4 present our main results: the theoretical steady-state MSEs for the SLA and the ACLA are respectively derived in Section 3 and Section 4. Section 5 provides simulation results to demonstrate the validity of the theoretical MSEs.
2. Preliminaries

The block diagram of the notch filter considered in this paper is shown in Fig. 1. Here, \( u(n) \) and \( y(n) \) are respectively the input signal and the output signal of the notch filter, and \( u(n) \) consists of a single sinusoid \( s(n) \) and a noise \( v(n) \) as follows:

\[
\begin{align*}
  u(n) &= s(n) + v(n) \quad (1) \\
  s(n) &= A \cos(\omega_s n + \phi) \quad (2)
\end{align*}
\]

where \( A \) and \( \omega_s \) respectively denote the amplitude and the unknown angular frequency of the sinusoid, and the initial phase \( \phi \) is a random variable that is uniformly distributed in \([0, 2\pi)\). The noise \( v(n) \) is assumed to be a zero-mean white Gaussian noise with variance \( \sigma_v^2 \) and uncorrelated to \( \phi \). The transfer function of the notch filter shown in Fig. 1(a) is

\[
H(z) = \frac{1}{2} [1 + H_{\text{AP}}(z)] \quad (3)
\]

where \( H_{\text{AP}}(z) \) is the second-order IIR all-pass function given by

\[
H_{\text{AP}}(z) = \frac{\sin \theta_2 + \sin \theta_1 (1 + \sin \theta_2) z^{-1} + z^{-2}}{1 + \sin \theta_1 (1 + \sin \theta_2) z^{-1} + \sin \theta_2 z^{-2}} \quad (4)
\]

and the block diagram of this all-pass filter is constructed by the normalized lattice structure with two lattice sections, as shown in Fig. 1(b) and (c). From (3) and (4), the transfer function of the notch filter is found to be

\[
H(z) = \frac{1 + \sin \theta_2}{2} \frac{1 + 2 \sin \theta_1 z^{-1} + z^{-2}}{1 + \sin \theta_1 (1 + \sin \theta_2) z^{-1} + \sin \theta_2 z^{-2}}. \quad (5)
\]

Hence this notch filter is based on the all-pass filter constructed by the normalized lattice structure \([1, 19, 30]\). The parameter \( \theta_1 \) controls the notch frequency \( \omega_0 \) and \( \theta_2 \) controls the 3-dB attenuation bandwidth \( B \) according to

\[
\begin{align*}
  \theta_1 &= \omega_0 - \frac{\pi}{2} \quad (6) \\
  \sin \theta_2 &= \frac{1 - \tan(B/2)}{1 + \tan(B/2)} \quad (7)
\end{align*}
\]

As such, the all-pass-based notch filter can tune \( \omega_0 \) and \( B \) independently.

Throughout this paper, it is assumed that \( \theta_2 \) is fixed and only \( \theta_1 \) is controlled by an adaptive algorithm in such a manner that the notch frequency \( \omega_0 \) estimates the frequency \( \omega_s \) of the sinusoid.
2.1. Simplified Lattice Algorithm (SLA)

The SLA [1, 19] makes use of the following update formula to estimate \( \omega_s \):

\[
\theta_1(n+1) = \theta_1(n) - \mu y(n)x_1(n) \tag{8}
\]

where \( \mu \) is the step-size parameter and \( x_1(n) \) is the state variable that corresponds to the output of the delay element in front of the \( \theta_1 \)-section, as shown in Fig. 2.

In our MSE analysis, we need to know the transfer function from \( u(n) \) to \( x_1(n) \). Denoting this transfer function by \( F(z) \), we find from Fig. 2 that

\[
F(z) = \frac{\cos \theta_1(n) \cos \theta_2 z^{-1}}{1 + \sin \theta_1(n)(1 + \sin \theta_2)z^{-1} + \sin \theta_2 z^{-2}}. \tag{9}
\]

2.2. Affine Combination Lattice Algorithm (ACLA)

The ACLA [28] is based on the block diagram of Fig. 3 and the following update formula

\[
\theta_1(n+1) = \theta_1(n) - \mu y(n)\psi(n) \tag{10}
\]

where \( \psi(n) \) is given by

\[
\psi(n) = kx_1(n) + (1-k)\nabla y(n). \tag{11}
\]

The weight parameter \( k \) controls the convergence speed and is set to be \( k > 1 \). Using larger value of \( k \) gives faster convergence but it causes larger MSE in the frequency estimate. The quantity \( \nabla y(n) \) is the gradient signal \( \partial y(n)/\partial \theta_1(n) \). It is proved in [1] that the signal \( \nabla_y(n) \) is obtained as the output of the following system

\[
G(z) = 1 - H(z) \\
= \frac{1}{2} [1 - H_{AP}(z)]. \tag{12}
\]

where the input signal of \( G(z) \) is set to be \( x_1(n) \). Hence \( \nabla_y(n) \) is given as in Fig. 3. It is proved and demonstrated in [28] that the ACLA achieves faster convergence than the SLA.

The transfer function from \( u(n) \) to \( \psi(n) \), denoted by \( J(z) \), is found from Fig. 3 to be

\[
J(z) = kF(z) + (1-k)F(z)G(z). \tag{13}
\]

This transfer function will be also used in our MSE analysis.
3. Frequency estimation MSE for SLA

The frequency estimation error at time $n$ is defined by

$$\delta(n) = \hat{\omega}_s(n) - \omega_s$$

where $\hat{\omega}_s(n)$ is the frequency estimate given by the notch frequency of the all-pass-based adaptive notch filter, i.e.

$$\hat{\omega}_s(n) = \omega_0(n) = \theta_1(n) + \frac{\pi}{2}$$

Since the all-pass-based adaptive notch filtering gives unbiased frequency estimation, it follows that $E[\delta(n)] = 0$ at the steady-state. Hence the variance of the estimation error is equal to the MSE. In view of this, our goal is to derive the steady-state frequency estimation MSE $E[\delta^2(n)]$ in closed-form for the SLA and the ACLA. We will achieve this goal by means of the following steps.

1. Apply the method of [16, 22] to the SLA/ACLA and establish a linear time-invariant model that represents the process of generation of the frequency estimation error $\delta(n)$.
2. Divide the aforementioned model into two separate submodels. These submodels respectively correspond to two error signals $\delta_1(n)$ and $\delta_2(n)$ that are uncorrelated to each other.
3. Derive closed-form expressions of $E[\delta_1^2(n)]$ and $E[\delta_2^2(n)]$, and obtain the frequency estimation MSE $E[\delta^2(n)]$ as the sum of $E[\delta_1^2(n)]$ and $E[\delta_2^2(n)]$.

3.1. Step 1: Construction of error generation model

Following the aforementioned strategy, first we will construct a linear time-invariant model for generation of $\delta(n)$. The resultant model is shown in Fig. 4 (derivation of this model is given in Appendix A), where the system $Q(z)$ is given by

$$Q(z) = \frac{-\mu z^{-1}}{1 - az^{-1}}$$

$$a = 1 - \frac{1}{2} \mu A^2 \beta_1 \beta_2$$

$$\beta_1 = \frac{1 + \sin \theta_2}{1 - \sin \theta_2}$$

$$\beta_2 = \frac{\cos \theta_2}{1 - \sin \theta_2}.$$
This model has three input signals $v_H(n)$, $v_F(n)$ and $A\beta_2 \sin(\omega_s n + \phi)$, and the output signal $\delta(n+1)$ corresponds to the frequency estimation error. The input signal $v_H(n)$ is obtained by passing the white noise $v(n)$ through the notch filter $H(z)$, where the notch frequency $\omega_0$ is set to be $\omega_0 = \omega_s$ (i.e. $\theta_1 = \omega_s - \pi/2$). Similarly, $v_F(n)$ is obtained by passing $v(n)$ through $F(z)$ with $\omega_0 = \omega_s$. From this setup, the input of the system $Q(z)$, denoted by $p(n)$, is found to be

$$p(n) = p_1(n) + p_2(n)$$

$$p_1(n) = A\beta_2 v_H(n) \sin(\omega_s n + \phi)$$

$$p_2(n) = v_H(n)v_F(n).$$

When this model attains steady-state, the frequency estimation MSE is obtained with the help of the Parseval’s theorem as follows:

$$E[\delta^2(n)] = E[\delta^2(n+1)] = \frac{1}{2\pi} \int_{-\pi}^{\pi} |Q(e^{j\omega})|^2 P_{pp}(e^{j\omega}) d\omega$$

where $P_{pp}(e^{j\omega})$ is the power spectral density function of $p(n)$. In this way, the frequency estimation MSE can be analytically described. However, this description is not given in closed-form and hence further investigation is necessary to obtain a closed-form expression.

Such a closed-form expression can be derived by means of conventional methods. For example, applying the method given in [24] to this model allows us to derive a closed-form expression of the MSE for the SLA. The derivation is achieved by further simplification of the model of Fig. 4 through the following approximation

$$v_F(n) \approx 0.$$ 

This approximation is based on the assumption that the 3-dB attenuation bandwidth $B$ is sufficiently small. However, the closed-form MSE expression that is derived by this approximation is not so accurate because this approximation is not valid for the SLA: as will be demonstrated in Section 5, in most cases $v_F(n)$ has relatively high contribution to the frequency estimation error.
3.2. Step 2: Division of error generation model

In order to derive a closed-form MSE expression with high accuracy, we divide the error generation model of Fig. 4 into two submodels, as shown in Fig. 5. These two submodels are respectively related to generation of $\delta_1(n+1)$ and $\delta_2(n+1)$. These two estimation errors are uncorrelated to each other and satisfy the following relationship:

$$ E[\delta^2(n)] = E[\delta_1^2(n)] + E[\delta_2^2(n)]. \quad (25) $$

Therefore, it suffices to derive closed-form expressions of $E[\delta_1^2(n)]$ and $E[\delta_2^2(n)]$ in order to achieve our goal.

In the rest of this subsection we will prove (25). First, consider the signal $p(n) = p_1(n) + p_2(n)$ in the original model of Fig. 4 and let $r_{pp}(m)$ be the autocorrelation function of $p(n)$. This autocorrelation function is expressed by

$$ r_{pp}(m) = E[p(n)p(n+m)] = r_{p_1p_1}(m) + r_{p_1p_2}(m) + r_{p_2p_1}(m) + r_{p_2p_2}(m) $$

The cross-correlation function $r_{p_1p_2}(m)$ is found to be

$$ r_{p_1p_2}(m) = E[p_1(n)p_2(n+m)] = E[v_H(n)A\beta_2 \sin(\omega_s n + \phi)v_F(n+m)] = 0. \quad (27) $$

In a similar way, $r_{p_2p_1}(m)$ is shown to be zero. Hence it follows that $p_1(n)$ and $p_2(n)$ are uncorrelated to each other and $r_{pp}(m)$ becomes

$$ r_{pp}(m) = r_{p_1p_1}(m) + r_{p_2p_2}(m). \quad (28) $$

Considering the discrete-time Fourier transform of (28), we have

$$ P_{pp}(e^{j\omega}) = P_{p_1p_1}(e^{j\omega}) + P_{p_2p_2}(e^{j\omega}) \quad (29) $$
where \( P_{pp}(e^{j\omega}), P_{p_1p_1}(e^{j\omega}) \) and \( P_{p_2p_2}(e^{j\omega}) \) respectively denote the power spectral density function of \( p(n) \), \( p_1(n) \) and \( p_2(n) \). Substituting (29) into (23) yields

\[
E[\delta^2(n)] = \frac{1}{2\pi} \int_{-\pi}^{\pi} |Q(e^{j\omega})|^2 P_{p_1p_1}(e^{j\omega}) d\omega + \frac{1}{2\pi} \int_{-\pi}^{\pi} |Q(e^{j\omega})|^2 P_{p_2p_2}(e^{j\omega}) d\omega. \tag{30}
\]

Now, let \( \delta_1(n+1) \) and \( \delta_2(n+1) \) be the signals as shown in the submodels of Fig. 5. Then, the following relationships hold:

\[
E[\delta_1^2(n)] = \frac{1}{2\pi} \int_{-\pi}^{\pi} |Q(e^{j\omega})|^2 P_{p_1p_1}(e^{j\omega}) d\omega \tag{31}
\]

\[
E[\delta_2^2(n)] = \frac{1}{2\pi} \int_{-\pi}^{\pi} |Q(e^{j\omega})|^2 P_{p_2p_2}(e^{j\omega}) d\omega. \tag{32}
\]

Substituting (31) and (32) into (30) yields (25) and completes the proof.

3.3. Step 3: Closed-form expressions of \( E[\delta_1^2(n)] \), \( E[\delta_2^2(n)] \) and \( E[\delta^2(n)] \)

Based on the submodels given in the previous step, we will describe the frequency estimation MSE in closed-form.

3.3.1. Derivation of \( E[\delta_1^2(n)] \)

It readily follows that the submodel of Fig. 5(a) becomes the same as the original model with \( v_{F}(n) = 0 \). As explained in Section 3.1, derivation of the MSE from the original model with \( v_{F}(n) = 0 \) is achieved using the method of [24]. Therefore, we will follow [24] to derive the closed-form expression of \( E[\delta_1^2(n)] \).

First, consider the autocorrelation function \( r_{p_1p_1}(m) \) that is calculated from (21) as follows:

\[
\begin{align*}
    r_{p_1p_1}(m) & = E[p_1(n)p_1(n + m)] \\
    & = A^2 \beta_2^2 E[v_H(n)v_H(n + m)] E[\sin(\omega_s n + \phi) \sin(\omega_s(n + m) + \phi)] \\
    & = \frac{A^2 \beta_2^2}{2} r_{v_Hv_H}(m) \cos(\omega_s m) \tag{33}
\end{align*}
\]

where \( r_{v_Hv_H}(m) = E[v_H(n)v_H(n + m)] \) is the autocorrelation function of \( v_H(n) \). From (33) the power spectral density function of \( p_1(n) \), denoted by
where $P_{v_Hv_H}(e^{j\omega})$ is the power spectral density function of $v_H(n)$. Using this expression and the Parseval’s theorem, we have

$$E[\delta_1^2(n)] = \frac{A^2 \beta_2^2}{2} \frac{1}{2\pi} \int_{-\pi}^{\pi} |Q(e^{j\omega})|^2 (P_{v_Hv_H}(e^{j(\omega-\omega_s)}) + P_{v_Hv_H}(e^{j(\omega+\omega_s)})) d\omega. \quad (35)$$

It should be noted that (35) can be rewritten as

$$E[\delta_1^2(n)] = \frac{A^2 \beta_2^2}{4} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( |Q(e^{j(\omega+\omega_s)})|^2 + |Q(e^{j(\omega-\omega_s)})|^2 \right) P_{v_Hv_H}(e^{j\omega}) d\omega \quad (36)$$

where the second line of (36) follows from $P_{v_Hv_H}(e^{j\omega}) = P_{v_Hv_H}(e^{-j\omega})$ and

$$\int_{-\pi}^{\pi} |Q(e^{j(\omega+\omega_s)})|^2 P_{v_Hv_H}(e^{j\omega}) d\omega = \int_{-\pi}^{\pi} |Q(e^{j(-\omega+\omega_s)})|^2 P_{v_Hv_H}(e^{-j\omega})(-d\omega) = \int_{-\pi}^{\pi} |Q(e^{j(\omega-\omega_s)})|^2 P_{v_Hv_H}(e^{j\omega}) d\omega. \quad (37)$$

Equation (36) means that $E[\delta_1^2(n)]$ is equal to the mean square of the output signal of the modified model shown in Fig. 6, i.e.

$$E[\delta_1^2(n)] = E[\delta_1'^2(n)]. \quad (38)$$

Based on this fact, in the sequel we consider this modified model and attempt to derive the closed-form expression of $E[\delta_1'^2(n)]$.

The derivation makes use of the state-space analysis. It is well known that the state-space representation can explicitly describe not only the input/output relationship but also the outputs of delay elements in a digital filter. Since the state variable used in the SLA (and the ACLA) corresponds to the output of the delay element in front of the lattice section of $\theta_1$, the state-space representation is very suitable to analysis of the behavior of the
state variable used in the SLA/ACLA, leading to explicit formulation of the MSE in closed-form.

In the state-space analysis, we first consider the system $H(z)$ of Fig. 6(a) in terms of the state-space representation as follows:

$$
\xi(n+1) = A\xi(n) + bv(n) \quad v_H(n) = c\xi(n) + dv(n)
$$

(39)

where $\xi(n) = (\xi_1(n) \quad \xi_2(n))^T$ is the state vector of which elements $\xi_1(n)$ and $\xi_2(n)$ are given as in Fig. 6(b). From Figs. 6(b) and 1(c), the coefficients $A, b, c$ and $d$ in (39) are respectively found to be

$$
A = \begin{pmatrix}
-\sin \theta_1 & -\cos \theta_1 \sin \theta_2 \\
\cos \theta_1 & -\sin \theta_1 \sin \theta_2
\end{pmatrix}, \quad b = \begin{pmatrix}
\cos \theta_1 \cos \theta_2 \\
\sin \theta_1 \cos \theta_2
\end{pmatrix}
$$

$$
c = \begin{pmatrix}
0 \\
1 + \sin \theta_2
\end{pmatrix}, \quad d = \frac{1 + \sin \theta_2}{2}.
$$

(40)

Next, consider the difference equation with respect to $\delta_1'(n)$, which is derived from (16) and Fig. 6 by

$$
\delta_1'(n+1) = a_{\delta}\delta_1'(n) + b_{\delta}v_H(n)
$$

(41)

where $a_{\delta}$ and $b_{\delta}$ are respectively given by

$$
a_{\delta} = ae^{j\omega}, \\
b_{\delta} = -\frac{1}{\sqrt{2}}\mu A\beta_2 e^{j\omega}.
$$

(42)

These results enable us to describe the relationship between $\delta_1'(n)$ and $v(n)$ in the following state-space form with the new state vector $\eta(n) \in \mathbb{R}^{3 \times 1}$:

$$
\eta(n+1) = \hat{A}\eta(n) + \hat{b}v(n) \\
\delta_1'(n+1) = \hat{c}\hat{A}\eta(n) + \hat{c}\hat{b}v(n)
$$

(43)

where $\eta(n), \hat{A}, \hat{b}$ and $\hat{c}$ are given by

$$
\eta(n) = \begin{pmatrix}
\xi(n) \\
\delta_1'(n)
\end{pmatrix}, \quad \hat{A} = \begin{pmatrix}
A & 0_{2 \times 1} \\
b_{\delta}c & a_{\delta}
\end{pmatrix}
$$

$$
\hat{b} = \begin{pmatrix}
b \\
b_{\delta}d
\end{pmatrix}, \quad \hat{c} = \begin{pmatrix}
o_{1 \times 2} & 1
\end{pmatrix}
$$

(44)
Here, $0_{m \times n}$ denotes the zero matrix with the size of $m \times n$. It is clear from (44) that $\delta'_1(n)$ appears as the third element of the state vector $\eta(n)$. Therefore, $E[\delta'^2_1(n)]$ corresponds to the (3, 3)-element of the state covariance matrix $K$ that is obtained as the solution to the following Lyapunov equation:

$$
K = E[\eta(n)\eta^H(n)] = \tilde{A}K\tilde{A}^H + \tilde{b}\tilde{b}^H \sigma_v^2.
$$

(45)

As a result, we obtain $E[\delta'^2_1(n)]$ (and $E[\delta'^2_2(n)]$) in terms of the following closed-form expression:

$$
E[\delta'^2_1(n)] = E[\delta'^2_2(n)] = \mu^2 A^2 \beta^2 \sigma_v^2 \frac{1 + \sin \theta_2}{1 - a^2} \left( 1 + \frac{1 - \sin \theta_2 \cos \omega_s}{C} \right)
$$

$$
C = 2a \left[ \sin \theta_2 a^3 - \cos^2 \omega_s (1 + \sin \theta_2) a^2 + \{ \cos 2 \omega_s + \cos^2 \omega_s (1 + \sin \theta_2) \} a - \cos^2 \omega_s \right]
$$

$$
D = \sin^2 \theta_2 a^4 - 2 \sin \theta_2 \cos^2 \omega_s (1 + \sin \theta_2) a^3 + \{2 \cos 2 \omega_s \sin \theta_2 + \cos^2 \omega_s (1 + \sin \theta_2)^2\} a^2 - 2 \cos^2 \omega_s (1 + \sin \theta_2) a + 1.
$$

(46)

It readily follows from (46) that $E[\delta'^2_1(n)]$ depends on the step-size parameter, the notch bandwidth parameter, and all of the components of the input signal.

3.3.2. Derivation of $E[\delta'^2_2(n)]$

Based on Fig. 5(b), the difference equation with respect to $\delta_2(n)$ becomes

$$
\delta_2(n + 1) = a\delta_2(n) - \mu v_H(n)v_F(n)
$$

(47)

from which we obtain

$$
E[\delta'^2_2(n)] = a^2 E[\delta'^2_2(n)] - 2a\mu E[v_H(n)v_F(n)\delta_2(n)] + \mu^2 E[v_H^2(n)v_F^2(n)]
$$

$$
\approx a^2 E[\delta'^2_2(n)] - 2a\mu E[v_H(n)v_F(n)]E[\delta_2(n)] + \mu^2 E[v_H^2(n)v_F^2(n)]
$$

$$
= a^2 E[\delta'^2_2(n)] + \mu^2 E[v_H^2(n)v_F^2(n)]
$$

(48)

where the approximation employed here is based on the method of [4, 16] and the last equality holds from the following fact [1]:

$$
E[v_H(n)v_F(n)] = 0.
$$

(49)
At the steady-state $E[\delta_2^2(n + 1)] = E[\delta_2^2(n)]$ holds and hence we have

$$E[\delta_2^2(n)] = \frac{\mu^2}{1 - a^2} E[v_H^2(n)v_F^2(n)].$$  \hspace{1cm} (50)

Under the assumption that $v_H(n)$ and $v_F(n)$ have a joint Gaussian distribution, $E[v_H^2(n)v_F^2(n)]$ can be easily derived [4] as follows:

$$E[v_H^2(n)v_F^2(n)] = \sigma_{v_H}^2 \sigma_{v_F}^2 + E[v_H(n)v_F(n)]^2$$

$$= \sigma_{v_H}^2 \sigma_{v_F}^2.$$  \hspace{1cm} (51)

where $\sigma_{v_H}^2$ and $\sigma_{v_F}^2$ are respectively given by [1]

$$\sigma_{v_H}^2 = \frac{1 + \sin \theta_2}{2} \sigma_v^2$$

$$\sigma_{v_F}^2 = \sigma_v^2.$$  \hspace{1cm} (52) (53)

Consequently, the closed-form expression of $E[\delta_2^2(n)]$ is obtained as follows:

$$E[\delta_2^2(n)] = \frac{\mu^2}{1 - a^2} \frac{1 + \sin \theta_2}{2} \sigma_v^4.$$  \hspace{1cm} (54)

Note that $E[\delta_2^2(n)]$ is independent of $\omega_s$, i.e. the frequency of the input sinusoid.

3.3.3. Closed-form expression of frequency estimation MSE and comparison with conventional methods

Using the aforementioned results, we obtain the frequency estimation MSE for the SLA as $E[\delta^2(n)] = E[\delta_1^2(n)] + E[\delta_2^2(n)]$, where $E[\delta_1^2(n)]$ and $E[\delta_2^2(n)]$ are respectively given in closed-form by (46) and (54).

It should be noted that, just applying the conventional methods [16, 22, 24] to the SLA gives only the term $E[\delta_1^2(n)]$ because of the approximation of $v_F(n) \approx 0$. As stated earlier, this approximation is not valid in the case of the SLA, where $v_F(n)$ gives relatively high contribution to the estimation error. Hence just applying the conventional methods results in loss of accuracy of the MSE analysis. On the other hand, our method based on the division of the error generation model does not carry out this approximation, meaning that the contribution of $v_F(n)$ is taken into account. As a result, the MSE expression given by our method consists of the term $E[\delta_2^2(n)]$ as well as $E[\delta_1^2(n)]$, leading to more accurate MSE expression than the direct use of the
conventional methods [16, 22, 24]. Similar result will be also derived in the case of the ACLA. The high accuracy of our method will be demonstrated by simulations in Section 5.

**Remark 1.** In this paper it is assumed that the input signal consists of a white noise and a single sinusoid. Extension of our method to multiple sinusoids will be a very difficult task. In this case a cascade connection of second-order notch filter sections is used, where the noise becomes colored at all of the notch filter sections except for the first section. This fact means that the assumption of the white noise input is violated in the multiple sinusoids case, and that our MSE description will lose accuracy to some extent.

**Remark 2.** In our theory, the white noise $v(n)$ in the input signal is assumed to be of a Gaussian distribution. Otherwise it becomes difficult to assume that the two signals $v_H(n)$ and $v_F(n)$ have a joint Gaussian distribution and to describe $E[v_H^2(n)v_F^2(n)]$ explicitly. Therefore, it remains a future task to investigate other types of noise inputs.

### 4. Frequency estimation MSE for ACLA

As shown in (10), the ACLA differs from the SLA in that $\psi(n)$ is used instead of $x_1(n)$. Since the MSE analysis for the SLA makes use of the white noise contribution $v_F(n)$ of the signal $x_1(n)$, in the case of the ACLA we will take into account $v_J(n)$ that corresponds to the white noise contribution of $\psi(n)$. It follows that $v_J(n)$ is obtained by passing $v(n)$ through the system $J(z)$ given by (13) with $\omega_0 = \omega_s$.

Let the frequency estimation error for the ACLA be $\delta(n)$ that is defined by (14). The frequency estimation MSE for the ACLA will be derived in the same way as in the case of the SLA. First, we construct the error generation model as in Fig. 7. This model is the same as Fig. 4, except for the noise contribution $v_F(n)$ replaced by $v_J(n)$.

Following the second step as in Section 3.2, we next divide this model into two submodels. The resultant submodel for $\delta_1(n)$ is the same as Fig. 5(a) and the submodel for $\delta_2(n)$ is shown in Fig. 8. The signals $p_1(n)$ and $p_2(n)$ in these submodels are uncorrelated to each other because it is shown
that

\[
E[p_1(n)p_2(n + m)] \\
= E[v_H(n)A\beta_2 \sin(\omega_s n + \phi)v_H(n + m)v_J(n + m)] \\
= E[A\beta_2 \sin(\omega_s n + \phi)|E[v_H(n)v_H(n + m)v_J(n + m)] \\
= 0
\]  

(55)

and \(E[p_2(n)p_1(n + m)]\) is shown to be zero in a similar way. Therefore the MSE is expressed as the sum of \(E[\delta_1^2(n)]\) and \(E[\delta_2^2(n)]\), i.e.

\[
E[\delta^2(n)] = E[\delta_1^2(n)] + E[\delta_2^2(n)].
\]  

(56)

Finally, we will derive \(E[\delta_1^2(n)]\) and \(E[\delta_2^2(n)]\) in closed-form. \(E[\delta_1^2(n)]\) is given by (46) because the submodel for \(\delta_1(n)\) for the ACLA is the same as that for the SLA. In order to derive \(E[\delta_2^2(n)]\), we start with

\[
\delta_2(n + 1) = a\delta_2(n) - \mu v_H(n)v_J(n)
\]  

(57)

from which we have

\[
E[\delta_2^2(n + 1)] \\
= a^2 E[\delta_2^2(n)] - 2a\mu E[v_H(n)v_J(n)\delta_2(n)] + \mu^2 E[v_H^2(n)v_J^2(n)] \\
\approx a^2 E[\delta_2^2(n)] - 2a\mu E[v_H(n)v_J(n)]E[\delta_2(n)] + \mu^2 E[v_H^2(n)v_J^2(n)] \\
= a^2 E[\delta_2^2(n)] + \mu^2 E[v_H^2(n)v_J^2(n)]
\]  

(58)

where the approximation follows from [4, 16] and the following property:

\[
E[v_H(n)v_J(n)] = 0.
\]  

(59)

Since \(E[\delta_2^2(n + 1)] = E[\delta_2^2(n)]\) holds at the steady-state, \(E[\delta_2^2(n)]\) becomes

\[
E[\delta_2^2(n)] = \frac{\mu^2}{1 - a^2} E[v_H^2(n)v_J^2(n)].
\]  

(60)

Here, under the assumption that \(v_H(n)\) and \(v_J(n)\) have a joint Gaussian distribution, \(E[v_H^2(n)v_J^2(n)]\) is found to be

\[
E[v_H^2(n)v_J^2(n)] = \sigma_{v_H}^2 \sigma_{v_J}^2 + E[v_H(n)v_J(n)]^2 \\
= \sigma_{v_H}^2 \sigma_{v_J}^2
\]  

(61)
where $\sigma^2_{vH}$ is given by (52) and $\sigma^2_{vJ}$ becomes (see Appendix B)

$$\sigma^2_{vJ} = \frac{1 + k^2}{2} \sigma^2_v. \quad (62)$$

From (52), (60), (61) and (62), $E[\delta^2(n)]$ is described in closed-form as

$$E[\delta^2_2(n)] = \frac{\mu^2}{1 - a^2} \frac{1 + \sin\theta_2}{2} \frac{1 + k^2}{2} \sigma^4_v. \quad (63)$$

It follows that (63) is independent of $\omega_s$, as is also the case of (54). However (63) differs from (54) in that the weight parameter $k$ for the ACLA is included. It is easy to see that (63) coincides with (54) if $k = 1$ holds.

Now we conclude that the frequency estimation MSE for the ACLA is obtained as $E[\delta^2(n)] = E[\delta^2_1(n)] + E[\delta^2_2(n)]$, with $E[\delta^2_1(n)]$ and $E[\delta^2_2(n)]$ respectively given by (46) and (63). From this result we see that the weight parameter $k$ in the ACLA appears in only $E[\delta^2_2(n)]$, and hence $E[\delta^2_1(n)]$ is independent of $k$.

**Remark 3.** Although this paper focuses on the MSE analysis for the SLA and ACLA, it is also important to theoretically evaluate and compare these algorithms from the viewpoints of other performances such as the dynamics, steady-state properties and step-size upper bounds. Since this topic has been studied in our previous work [28], we shall not go into the details in this paper.

### 5. Simulations and Discussions

The simulation examples given in this section show comparisons between the theoretical values of the frequency estimation MSE (i.e. $E[\delta^2(n)] = E[\delta^2_1(n)] + E[\delta^2_2(n)]$), which is based on our proposed method, and the simulated values. In order to confirm the effect of the term $E[\delta^2_2(n)]$, we also compare our theoretical MSEs with the conventional theoretical MSEs that correspond to $E[\delta^2(n)] = E[\delta^2_1(n)]$ and are derived by the direct use of [16, 22, 24].

In all the simulations given here, the initial notch frequency for the SLA and the ACLA is set to be $\hat{\omega}_s(0) = \pi/2$ radians. Also, in order to obtain the simulated values of the steady-state MSE, we calculate the average of $\delta^2(n)$ in the region of $1.5 \times 10^6 < n \leq 2 \times 10^6$, and we further take its ensemble average over 20 independent runs.
5.1. SLA

The simulation results for the SLA are shown in Fig. 9. Fig. 9(a) shows the MSE comparison with respect to the 3-dB attenuation bandwidth parameter \( \sin \theta_2 \), where the amplitude and the frequency of the input sinusoid are respectively \( A = 1 \) and \( \omega_s = 0.4\pi \) radians. The step-size parameter \( \mu \) is set to be \( \mu = 10^{-5} \). Also, the comparison is performed for two cases of input Signal-to-Noise Ratio (SNR), i.e. SNR = 0 dB and SNR = 10 dB. The input SNR is defined by

\[
\text{SNR} = 10 \log_{10} \frac{A^2}{2\sigma^2}[\text{dB}].
\]  

Fig. 9(b) is the MSE comparison with respect to the frequency \( \omega_s \) of the input sinusoid, where the other parameters are set to be \( A = 1 \), \( \mu = 10^{-5} \), and SNR = 10 dB. Here, two cases are considered for the 3-dB attenuation bandwidth parameter: the case of narrow bandwidth (\( \sin \theta_2 = 0.96 \)) and the case of relatively wide bandwidth (\( \sin \theta_2 = 0.88 \)). Fig. 9(c) shows the MSE comparison with respect to the step-size parameter \( \mu \), where the other parameters are \( A = 1 \), \( \omega_s = 0.4\pi \) radians, and \( \sin \theta_2 = 0.9 \). For the input SNR, two cases are considered: SNR = 0 dB and SNR = 10 dB.

In all of these simulation results, our theoretical result, corresponding to \( E[\delta_1^2(n)] + E[\delta_2^2(n)] \), shows very good agreement with the simulated values of the MSE. On the other hand, if we use only \( E[\delta_1^2(n)] \) for the MSE by means of the conventional methods, the theoretical value deviates from the simulated values. This fact shows that the term \( E[\delta_2^2(n)] \) highly contributes to the accuracy of the MSE analysis. In particular, we can see the high contribution of \( E[\delta_2^2(n)] \) to the MSE in the cases of wide notch bandwidth and low input SNR. In these cases, the contribution of \( v_F(n) \) to the state variable \( x_1(n) \) in the notch filter becomes relatively large, meaning that the ratio of \( E[\delta_2^2(n)] \) to \( E[\delta^2(n)] \) also becomes large. Hence the term \( E[\delta_2^2(n)] \) has large effect on the MSE.

5.2. ACLA

The simulation results for the ACLA are shown in Fig. 10. The simulation given in Fig. 10(a) is the MSE comparison with respect to the notch bandwidth parameter \( \sin \theta_2 \), where the weight parameter \( k \) is set to be \( k = 2 \) and the other parameters are the same as in Fig. 9(a). The MSE comparison with respect to the frequency \( \omega_s \) of the input sinusoid is shown in Fig. 10(b),
where \( k = 2 \) and the other parameters are the same as in Fig. 9(b). The comparison with respect to \( \mu \) is shown in Fig. 10(c), where \( k = 2 \) and the other parameters are the same as in Fig. 9(c). Finally, the MSE comparison with respect to the weight parameter \( k \) is shown in Fig. 10(d), where \( A = 1, \omega_s = 0.4\pi \) radians, \( \sin \theta_2 = 0.9 \) and \( \mu = 10^{-5} \), and the input SNR is set to be \( \text{SNR} = 0 \) dB and \( \text{SNR} = 10 \) dB.

In the case of the ACLA as well as the SLA, it follows that our theoretical result shows very good agreement with the simulated values, and that the term \( E[\delta^2_k(n)] \) gives high contribution to the accuracy of the MSE analysis. In addition, Fig. 10(d) shows that the dependency of \( k \) on the MSE becomes apparent by introducing the term \( E[\delta^2_k(n)] \). This result coincides with our theoretical result for the ACLA given in Section 4.

5.3. Comparison with other types of adaptive notch filters

Some readers may consider that the MSEs for the SLA/ACLA should be compared with the ones for other types of adaptive notch filters such as the constrained poles and zeros and the bilinear IIR notch filters. We consider that a fair comparison of MSEs between different types of adaptive notch filters is a difficult task because the transfer functions used in these notch filters are different. The SLA and the ACLA are based on the all-pass-based notch filter, which makes use of a different transfer function from the one based on the constrained poles and zeros or the bilinear notch filter. It is apparent that, if two adaptive notch filtering techniques to be compared are given by different transfer functions, then their performances also show different behavior. For example, as is proved in [31], the output power due to the white noise input for the all-pass-based notch filter becomes different from the one for the constrained poles and zeros, even if both notch filters have the same notch bandwidth. This means that the output SNRs of these two adaptive notch filters become different, which makes a fair performance comparison among different types of adaptive notch filters a difficult task.

However, in the case where the output SNR is not so important and the frequency estimation is a main issue, it is possible to carry out a fair comparison of MSEs between different adaptive notch filters. Therefore in this subsection we shall give an MSE comparison between the SLA/ACLA and the adaptive notch filter based on the constrained poles and zeros. Although there exist many frequency estimation algorithms using the constrained poles and zeros, in this paper we pay attention to the plain gradient algorithm [4, 5] because the steady-state MSE is well analyzed for this algorithm. Here we
refer to this method as CPZ-PG (Constrained Poles and Zeros with Plain Gradient algorithm) and we compare the steady-state MSE for the CPZ-PG with that for the SLA/ACLA.

Fig. 11 shows the result of the MSE comparison between the CPZ-PG, the SLA, and the ACLA. Here, the MSE is compared with respect to the input SNR in order to address the Cramer-Rao lower bound of which details will be discussed in the next subsection. For a fair comparison, the same parameters are used for all of the methods shown here: the step-size parameter and the notch bandwidth parameter are respectively set to be $\mu = 10^{-5}$ and $\sin \theta_2 = \rho = 0.9$, where $\rho$ is the pole radius that determines the notch bandwidth of the CPZ-based notch filter\(^1\). The weight parameter $k$ for the ACLA is set to be $k = 2$, and the simulation setups such as the initial notch frequency and the data samples used for calculation of the MSE are the same as in the previous subsections. From Fig. 11 it is found that the SLA and the ACLA outperform the CPZ-PG in terms of the MSE, under the same step-size parameter and the notch bandwidth. The reason of this result is due to the fact that the CPZ-based adaptive notch filter generates the bias in the frequency estimate. If other frequency estimation algorithms with bias removal are used for the CPZ-based adaptive notch filter, their frequency estimation MSEs may outperform the SLA/ACLA. However such investigation is beyond the scope of this paper and omitted here.

5.4. Comparison with Cramer-Rao Lower Bound

The Cramer-Rao Lower Bound (CRLB) is well known as one of the tools of evaluation of frequency estimation accuracy for sufficiently large data sets. According to [32], the CRLB on the asymptotic variance of the frequency estimation error is given by

$$CRLB = \frac{1}{L^3} \frac{24\sigma_v^2}{A^2}$$  \hspace{1cm} (65)

where $L$ denotes the number of data samples of a given input signal. From (65) we know that the lower bound of the frequency estimation MSE depends on $L$ and the input SNR.

\(^1\)Note that the pole radius of the all-pass-based notch filter is given by $\sin \theta_2$. Hence the setup of $\sin \theta_2 = \rho = 0.9$ means that the same pole radius (and the same notch bandwidth) is used for the CPZ-PG, the SLA, and the ACLA.
Now, a comparison between the CRLB and the error variance for the CPZ-PG, the SLA, and the ACLA is shown as in Fig. 12. For simplicity, the error variance for each method is shown in terms of only the theoretical value. Note that the error variance for the CPZ-PG becomes lower than its MSE because of the existence of the estimation bias. The CRLB shown here is based on $L = 1.5 \times 10^6$, according to the simulation setups given in the previous subsections. We clearly see from Fig. 12 that the CRLB is much lower than the error variance for the CPZ-PG/SLA/ACLA. Hence further investigation will be needed for improvement of all of these adaptive notch filtering methods, in order to attain their estimation errors closer to the CRLB.

6. Conclusion

This paper has derived the frequency estimation MSE in closed-form for two algorithms in the all-pass-based adaptive notch filter with normalized lattice structure, i.e. the SLA and the ACLA. The derivation is based on construction of a linear time-invariant model for generation of the frequency estimation error, and division of this model into two submodels of which output signals are uncorrelated to each other. This model-division strategy has led to more accurate MSE expressions than directly using the existing analysis methods. The simulation results have demonstrated that our theoretical MSE expressions agree very well with the simulated values.

Future tasks include investigations of the MSE analysis for the cases of multiple sinusoids and different types of noises. Also, extension of our MSE analysis to the complex-valued adaptive notch filtering with the normalized lattice structure, such as the method presented in [33], is a topic for future research.

Acknowledgment

This work was supported by JSPS KAKENHI Grant Number 15K06049.

Appendix A. Derivation of error generation model

We start with description of the relationship between $\delta(n)$ and $\delta(n + 1)$. This is easily obtained from (8), (14) and (15) as follows:

$$\delta(n + 1) = \delta(n) - \mu y(n)x_1(n).$$

(A.1)
Since the input signal consists of the sinusoid $s(n)$ and the additive white noise $v(n)$, the signals $y(n)$ and $x_1(n)$ in (A.1) are expressed by

\[
\begin{align*}
y(n) & = s_H(n) + v_H(n) \\
x_1(n) & = s_F(n) + v_F(n)
\end{align*}
\]  

where $s_H(n)$ and $v_H(n)$ are respectively obtained by passing $s(n)$ and $v(n)$ through the notch filter $H(z)$ with $\omega_0 = \omega_s$. Similarly, $s_F(n)$ and $v_F(n)$ are respectively obtained by passing $s(n)$ and $v(n)$ through the system $F(z)$ with $\omega_0 = \omega_s$.

Next we will describe $s_H(n)$ and $s_F(n)$. Since $s(n)$ is the single sinusoid with the frequency of $\omega_s$, $s_H(n)$ and $s_F(n)$ are respectively obtained from $H(z)$ and $F(z)$ with $z = e^{j\omega_s}$. These frequency responses are respectively found to be

\[
\begin{align*}
H(e^{j\omega_s}) & = \frac{1 + \sin \theta_2}{2} \frac{1 + 2 \sin \theta_1(n) e^{-j\omega_s} + e^{-2j\omega_s}}{1 + \sin \theta_1(n)(1 + \sin \theta_2)e^{-j\omega_s} + \sin \theta_2 e^{-2j\omega_s}} \\
& = \frac{1 + \sin \theta_2}{2} \frac{1 - 2 \cos(\delta(n) + \omega_s)e^{-j\omega_s} + e^{-2j\omega_s}}{1 - \cos(\delta(n) + \omega_s)(1 + \sin \theta_2)e^{-j\omega_s} + \sin \theta_2 e^{-2j\omega_s}} \\
& \approx -j\beta_1 \delta(n) \\
F(e^{j\omega_s}) & = \frac{\cos \theta_1(n) \cos \theta_2 e^{-j\omega_s}}{1 + \sin \theta_1(n)(1 + \sin \theta_2)e^{-j\omega_s} + \sin \theta_2 e^{-2j\omega_s}} \\
& \approx -j\beta_2
\end{align*}
\]

where $\beta_1$ and $\beta_2$ are respectively given by (18) and (19). In (A.4), the second equality follows from (14) and (15), and the approximation used here is based on the first-order Taylor series expansion around $\delta(n) = 0$ under the assumption of $|\delta(n)| \ll 1$. The approximation used in (A.5) is based on $\tilde{\omega}_s(n) \approx \omega_s$ under the assumption of sufficiently slow adaptation. The result of this approximation gives the first term of the Taylor series expansion of $F(e^{j\omega_s})$ with $\theta_1(n) = \delta(n) + \omega_s - \pi/2$ around $\delta(n) = 0$. From (A.4), (A.5) and (2), the signals $s_H(n)$ and $s_v(n)$ are respectively approximated to be

\[
\begin{align*}
s_H(n) & \approx \delta(n) A_{\beta_1} \sin(\omega_s n + \phi) \\
s_F(n) & \approx A_{\beta_2} \sin(\omega_s n + \phi)
\end{align*}
\]
from which $y(n)$ and $x_1(n)$ are described by

$$y(n) \approx \delta(n) A \beta_1 \sin(\omega_s n + \phi) + v_H(n) \quad (A.8)$$
$$x_1(n) \approx A \beta_2 \sin(\omega_s n + \phi) + v_F(n). \quad (A.9)$$

Substituting (A.8) and (A.9) into (A.1) yields the difference equation with respect to the frequency estimation error $\delta(n)$ as follows:

$$\delta(n + 1) = \left\{1 - \mu A^2 \beta_1 \beta_2 \sin^2(\omega_s n + \phi) - \mu A \beta_2 \sin(\omega_s n + \phi) v_F(n)\right\} \delta(n)$$
$$- \mu \left\{A \beta_2 v_H(n) \sin(\omega_s n + \phi) + v_H(n) v_F(n)\right\}.$$

(A.10)

Now, in order to simplify the analysis, we assume that the time variation of $\delta(n)$ is sufficiently smaller than that of the input signal and approximate (A.10) as follows:

$$\delta(n + 1) \approx \left(1 - \frac{1}{2} \mu A^2 \beta_1 \beta_2 \right) \delta(n)$$
$$- \mu \left\{A \beta_2 v_H(n) \sin(\omega_s n + \phi) + v_H(n) v_F(n)\right\}.$$

(A.11)

This difference equation gives the model of Fig. 4.

Appendix B. Derivation of $\sigma^2_{v_J}$

In deriving $\sigma^2_{v_J}$, we make frequent use of the inner product that is defined for two given transfer functions $X(z)$ and $Y(z)$ as follows:

$$\langle X(z), Y(z) \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) Y(e^{-j\omega}) d\omega. \quad (B.1)$$

Since $v_J(n)$ is the white noise contribution of the output signal of $J(z)$, its variance $\sigma^2_{v_J}$ is given by

$$\sigma^2_{v_J} = \sigma^2_{v_J} \langle J(z), J(z) \rangle$$
$$= \sigma^2_{v_J} \langle k F(z) + (1 - k) F(z) G(z), k F(z) + (1 - k) F(z) G(z) \rangle$$
$$= \sigma^2_{v_J} \left\{k^2 \langle F(z), F(z) \rangle + 2k(1 - k) \langle F(z), F(z) G(z) \rangle + (1 - k)^2 \langle F(z) G(z), F(z) G(z) \rangle \right\}. \quad (B.2)$$
The first term in the right-hand side of (B.2) becomes
\[ k^2 \langle F(z), F(z) \rangle = k^2 \] (B.3)
because \( \langle F(z), F(z) \rangle = 1 \) holds [1]. In order to calculate the inner product
\( \langle F(z), F(z)G(z) \rangle \) in the second term, we rewrite \( G(z) \) as
\[ G(z) = \frac{1}{2} (1 - V(z)) \] (B.4)
where \( V(z) \) is the second-order all-pass filter of the form
\[ V(z) = \frac{z^{-2}D(z^{-1})}{D(z)} \] (B.5)
\[ D(z) = 1 + \sin \theta_1 (1 + \sin \theta_2) z^{-1} + \sin \theta_2 z^{-2}. \] (B.6)
From these relationships, the inner product
\[ \langle F(z), F(z)G(z) \rangle \]
becomes
\[ \langle F(z), F(z)G(z) \rangle = \frac{1}{2} (1 - \langle F(z), F(z)V(z) \rangle) \] (B.7)
Note that the term \( \langle F(z), F(z)V(z) \rangle \) in (B.7) vanishes because
\[ \langle F(z), F(z)V(z) \rangle = \left\langle \frac{\cos \theta_1 \cos \theta_2 z^{-1}}{D(z)}, \frac{\cos \theta_1 \cos \theta_2 z^{-2}}{D(z)} \right\rangle \]
\[ = \left\langle \frac{\cos \theta_1 \cos \theta_2 D(z)}{D(z)}, \frac{\cos \theta_1 \cos \theta_2 z^{-2}}{D^2(z)} \right\rangle \]
\[ = \left\langle \frac{\cos \theta_1 \cos \theta_2}{D^2(z)} \right\rangle \]
\[ = 0. \] (B.8)
Hence the second term in the right-hand side of (B.2) becomes
\[ 2k(1 - k) \langle F(z), F(z)G(z) \rangle = k(1 - k). \] (B.9)
Finally, the third term in the right-hand side of (B.2) is found to be
\( (1 - k)^2 \langle F(z)G(z), F(z)G(z) \rangle \)
\[ = \frac{(1 - k)^2}{4} \langle F(z)(1 - V(z)), F(z)(1 - V(z)) \rangle \]
\[ = \frac{(1 - k)^2}{4} \left( \langle F(z), F(z) \rangle - 2 \langle F(z), F(z)V(z) \rangle + \langle F(z)V(z), F(z)V(z) \rangle \right) \]
\[ = \frac{(1 - k)^2}{2}. \] (B.10)
Substituting (B.3), (B.9) and (B.10) into (B.2), we obtain \( \sigma^2_{\nu} \) as in (62).

References


Figure 1: All-pass-based notch filter with normalized lattice structure: (a) block diagram of notch filter $H(z)$, (b) block diagram of all-pass filter $H_{AP}(z)$, and (c) block diagram of $i$-th lattice section.

Figure 2: Block diagram of SLA.
Figure 3: Block diagram of ACLA.

Figure 4: Linear time-invariant model for generation of frequency estimation error for SLA.
Figure 5: Division of error generation model: (a) submodel for generation of $\delta_1(n+1)$, and (b) submodel for generation of $\delta_2(n+1)$.

Figure 6: Modified model equivalent to the submodel for $\delta_1(n+1)$: (a) model for generation of $\delta'_1(n+1)$, and (b) relationship of $v(n)$, $v_H(n)$ and the state vector $\xi(n)$. 

$$A\beta_2 \sin(\omega_s n + \phi)$$

$$v_H(n) \xrightarrow{Q(z)} p_1(n) \xrightarrow{} \delta_1(n+1)$$

(a)

$$v_H(n) \xrightarrow{p_2(n)} Q(z) \xrightarrow{} \delta_2(n+1)$$

(b)

$$(a)$$

$$v(n) \xrightarrow{H(z)} \frac{A\beta_2}{\sqrt{2}} Q(ze^{-j\omega_s}) \xrightarrow{} \delta'_1(n+1)$$

(a)

$$v(n) \xrightarrow{v_H(n)} 1/2 + \sum_{i=2}^{\infty} \xi_i(n) \xrightarrow{\theta_1} \xi_1(n)$$

(b)

Figure 6: Modified model equivalent to the submodel for $\delta_1(n+1)$: (a) model for generation of $\delta'_1(n+1)$, and (b) relationship of $v(n)$, $v_H(n)$ and the state vector $\xi(n)$. 

30
Figure 7: Error generation model for ACLA.

Figure 8: Submodel for generation of \( \delta_2(n) \) for ACLA.
Figure 9: Simulation results for SLA: (a) 3-dB attenuation bandwidth versus MSE, (b) frequency of the sinusoid versus MSE, and (c) step-size parameter versus MSE.
Figure 10: Simulation results for ACLA: (a) 3-dB attenuation bandwidth versus MSE, (b) frequency of the sinusoid versus MSE, (c) step-size parameter versus MSE, and (d) weight parameter for ACLA versus MSE.
Figure 11: Input SNR versus MSE for CPZ-PG, SLA and ACLA.

Figure 12: Comparison between CRLB and error variance for CPZ-PG/SLA/ACLA.