Realization of Variable Low-Pass State-Space Digital Filters Using Step Responses

Shunsuke Koshita, Masahide Abe and Masayuki Kawamata
Department of Electronic Engineering, Graduate School of Engineering, Tohoku University
6-6-05, Aoba, Aramaki, Aoba-ku, Sendai, 980-8579 Japan
Email: kosita@mk.ecei.tohoku.ac.jp

Abstract—This paper presents a new realization method of variable low-pass infinite impulse response (IIR) digital filters with high tuning accuracy. We construct the proposed variable filters by using the state-space-based frequency transformation proposed by Mullis and Roberts. Since this transformation requires an inverse matrix that leads to high computational cost, we present a new simple algorithm that obtains this inverse matrix by using the step responses of a simple state-space system. This approach imposes no restrictions on the tuning parameter, and thus the cutoff frequencies of the proposed variable filters can be tuned arbitrarily. Furthermore, we show that the proposed variable filters exhibit high tuning accuracy with respect to quantization effects over the entire tuning range.

I. INTRODUCTION

Frequency selective digital filters with tunable characteristics are called variable digital filters and they are widely used in many applications of signal processing. One of the well-known methods for designing infinite-impulse response (IIR) variable digital filters is to make use of frequency transformation [1]: the transfer function of a variable IIR digital filter, denoted by $H_d(z)$, is obtained by the following transformation

$$H_d(z) = H_p(z)|_{z = e^{j\omega}} - T(z)$$ (1)

where $H_p(z)$ and $T(z)$ are the transfer functions of a prototype low-pass filter and an appropriate all-pass filter, respectively.

From the realization point of view, it is clear that the variable IIR digital filters designed as above should possess high tuning accuracy. Such high-accuracy realization can be achieved by using low-sensitivity structures for prototype digital filters. As is well-known, low-sensitivity digital filter structures can be obtained with the help of state-space representation [2]–[5], and thus the state-space representation will be attractive for realization of high-accuracy variable IIR digital filters. In fact, our previous work [6] presented variable low-pass IIR digital filters by using state-space representation, and we showed that this method outperforms the other existing methods with respect to the tuning accuracy.

However, the method of [6] can be applied to the limited class of transfer functions: the poles of the applicable transfer functions must be distinct complex conjugate pairs. In addition, the tuning range of the variable filters obtained by this method is also limited because this method applies linear approximation (truncation of the Taylor series expansion) [7] in order to reduce the complexity for calculation of the coefficients of $H_d(z)$.

In this paper, we propose a new method of realization of variable low-pass IIR digital filters in state-space form. Our new method is applicable to any stable transfer function. In addition, the proposed variable filters have no limitation on the tuning range. Furthermore, by choosing low-sensitivity structures for prototype state-space filters, the proposed variable filters possess very high accuracy with respect to quantization effects over the entire tuning range. We achieve these goals with the help of the state-space-based first-order frequency transformation that can preserve the controllability/observability Gramians of prototype filters [8]. Although this transformation requires an inverse matrix that leads to high computational cost, our proposed method avoids this problem by extending the method of [9] to the state-space-based frequency transformation: we obtain the required inverse matrix from the step responses of a simple state-space system. This approach greatly reduces the system complexity of the proposed variable filters without loss of tuning accuracy.

II. PRELIMINARIES

Let $H_p(z)$ be the transfer function of an $N$-th order prototype digital filter. This system is described by the following state-space equations

$$x_p(n + 1) = A_p x_p(n) + b_p u(n)$$ (2)

$$y(n) = c_p x_p(n) + d_p u(n)$$ (3)

where $u(n), y(n)$ and $x_p(n)$ are the scalar input, the scalar output and the state vector of size $N \times 1$, and $A_p, b_p, c_p$ and $d_p$ are real-valued coefficient matrices with appropriate size. The coefficients $(A_p, b_p, c_p, d_p)$ and the transfer function $H_p(z)$ are related as

$$H_p(z) = d_p + c_p(z I_N - A_p)^{-1} b_p$$ (4)

where $I_N$ is the $N \times N$ identity matrix. In this paper, the system $(A_p, b_p, c_p, d_p)$ is assumed to be a minimal realization of $H_p(z)$, i.e. the system $(A_p, b_p, c_p, d_p)$ is controllable and observable.

For the system $(A_p, b_p, c_p, d_p)$, the solutions $K_p$ and $W_p$ to the following Lyapunov equations are called the controlla-
bility Gramian and the observability Gramian, respectively:

\[ K_p = A_p K_p A_p^T + b_p b_p^T \]  \hspace{1cm} (5)
\[ W_p = A_p^T W_p A_p + c_p^T c_p. \]  \hspace{1cm} (6)

The Gramians \( K_p \) and \( W_p \) are symmetric and positive definite because the system \((A_p, b_p, c_p, d_p)\) is assumed to be asymptotically stable, controllable and observable.

It is well-known that these Gramians play important roles in realization of useful digital filter structures. For example, the balanced realization [10] that satisfies the relationship \( K_p = W_p = \Theta \), where \( \Theta \) is the diagonal matrix consisting of the positive square roots of the eigenvalues of the matrix product \( K_p W_p \), is known to be one of the low-sensitivity structures with respect to quantization effects. Moreover, this realization has another useful property that the structure is free of limit cycles [5].

Next we introduce a state-space formulation of the frequency transformation of (1). Here we concentrate on the low-pass-low-pass (LP-LP) transformation that uses the all-pass function \( T(z) \) defined as

\[ T(z) = \frac{z^{-1} - \eta}{1 - \eta z^{-1}}, \quad \eta = \frac{\sin \left( \frac{\omega_p - \omega_d}{2} \right)}{\sin \left( \frac{\omega_d}{2} \right)}, \quad |\eta| < 1 \]  \hspace{1cm} (7)

where \( \omega_p \) and \( \omega_d \) are the cutoff frequencies of the prototype filter \( H_p(z) \) and the desired variable low-pass filter \( H_d(z) \), respectively. In [8], a significant state-space representation of the above LP-LP transformation is presented: this representation gives the coefficients \((A_d, b_d, c_d, d_d)\) of the transfer function \( H_d(z) \) in terms of \((A_p, b_p, c_p, d_p)\) and \( \eta \) as follows:

\[ A_d = (\eta I_N + A_p)(I_N + \eta A_p)^{-1} \]  \hspace{1cm} (8)
\[ b_d = \sqrt{1 - \eta^2}(I_N + \eta A_p)^{-1}b_p \]  \hspace{1cm} (9)
\[ c_d = \sqrt{1 - \eta^2}c_p(I_N + \eta A_p)^{-1} \]  \hspace{1cm} (10)
\[ d_d = d_p - \eta c_p(I_N + \eta A_p)^{-1}b_p. \]  \hspace{1cm} (11)

An important property of the LP-LP transformation given by (8)–(11) is that the controllability/observability Gramians of the transformed filter, which are respectively denoted by \( K_d \) and \( W_d \), are the same as those of the prototype filter, i.e.

\[ K_d = K_p \]  \hspace{1cm} (12)
\[ W_d = W_p \]

for any \( \eta \). This property clearly shows that, if we construct the prototype filter \((A_p, b_p, c_p, d_p)\) with a useful structure such as the balanced realization, the transformed filter \((A_d, b_d, c_d, d_d)\) maintains the same structure as that of \((A_p, b_p, c_p, d_p)\).

III. MAIN RESULTS

Our proposed method applies the state-space-based frequency transformation given by (8)–(11) to realization of high-accuracy variable low-pass filters. A serious problem of this approach is that the direct use of (8)–(11) requires the computation of the inverse matrix \((I_N + \eta A_p)^{-1}\), which causes a very complicated system for tuning the frequency characteristics. One possible way to avoid this problem is to use a linear approximation by assuming the tuning parameter \( \eta \) to be close to zero. However, as stated earlier, such a linear approximation significantly limits the tuning range.

Instead, we present another efficient algorithm, where we obtain the required inverse matrix from the step responses of a specific state-space system. This algorithm is derived by extending the method of [9], which was originally proposed for direct-form transfer functions, to state-space representation. We now present the following theorem that will be used to easily obtain the required inverse matrix.

**Theorem 1:** Consider the following state-space system

\[ \Psi(k + 1) = -\eta A_p \Psi(k) + I_N u_s(k) \]  \hspace{1cm} (13)

where \( u_s(k) \) is the scalar unit step input and \( \Psi(k) \) is the output. Then, the following relationship holds:

\[ \lim_{k \to \infty} \Psi(k) = (I_N + \eta A_p)^{-1}. \]  \hspace{1cm} (14)

**Proof:** Taking the \( z \)-transform of (13), we have

\[ z \Psi(z) = -\eta A_p \Psi(z) + I_N \frac{1}{1 - z^{-1}} \]  \hspace{1cm} (15)

where we let \( \Psi(z) \) be the \( z \)-transform of \( \Psi(k) \) and we used the fact that the \( z \)-transform of \( u_s(k) \) is \( 1/(1 - z^{-1}) \). From the above equation we obtain

\[ \Psi(z) = (z I_N + \eta A_p)^{-1} \cdot \frac{1}{1 - z^{-1}}. \]  \hspace{1cm} (16)

Applying the final value theorem of the \( z \)-transform to (16) yields

\[ \lim_{k \to \infty} \Psi(k) = \lim_{z \to 1} (1 - z^{-1}) \Psi(z) \]
\[ = \lim_{z \to 1} (1 - z^{-1})(z I_N + \eta A_p)^{-1} \cdot \frac{1}{1 - z^{-1}} \]
\[ = (I_N + \eta A_p)^{-1}. \]  \hspace{1cm} (17)

Recalling \( |\eta| < 1 \) and the assumption that the prototype filter is stable, we easily see that the state-space system given by (13) is stable for any \( \eta \) and \( A_p \). Therefore, it follows from (17) that the step responses of this state-space system is guaranteed to converge to \((I_N + \eta A_p)^{-1}\), which completes the proof.

Theorem 1 shows that the required inverse matrix for the state-space-based frequency transformation can be obtained from the converged value of the step responses \( \Psi(k) \). Hence this approach does not require the division operation, and can be implemented by only the simple operations such as additions and multiplications. Furthermore, this approach does not impose any restriction on the tuning range, which will enable arbitrary tuning of the cutoff frequencies of our proposed variable filters.

We are now ready to present new variable low-pass filters with high tuning accuracy. From the above discussion it follows that the high-accuracy variable low-pass filters can be effectively constructed from (8)–(11), where the inverse matrix is replaced by the converged value of \( \Psi(k) \). Consequently,
the proposed variable low-pass filters are obtained through the following steps:

Step 1: Use the system given by (13) to calculate $\Psi(k_c)$, where $k_c$ is the number of iterations required for convergence of the step responses $\Psi(k)$.

Step 2: Using $\Psi(k_c)$, $(A_p, b_p, c_p, d_p)$ and $\eta$, construct the following realization $(A_d, b_d, c_d, d_d)$:

$$
A_d = (\eta I_N + A_p) \Psi(k_c) \tag{18}
$$

$$
b_d = \sqrt{1 - \eta^2} c_p \Psi(k_c) b_p \tag{19}
$$

$$
c_d = \sqrt{1 - \eta^2} c_p \Psi(k_c) \tag{20}
$$

$$
d_d = d_p - \eta c_p \Psi(k_c) b_p. \tag{21}
$$

This is the state-space representation of our proposed variable low-pass filters.

Step 3: If $\eta$ is updated to change the cutoff frequency, go back to Step 1 to recalculate $(A_d, b_d, c_d, d_d)$.

Here we discuss the significance of our proposed method. As is mentioned in Section I, our previous work [6] can be applied to the prototype filters with only the limited class of transfer functions. In contrast, our proposed method in this paper can be applied to any stable prototype filter because the transformation given by (18)–(21) is applicable to any stable realization $(A_p, b_p, c_p, d_p)$. Furthermore, in our previous work the tuning range is also limited due to the linear approximation, whereas our proposed method allows arbitrary tuning of the cutoff frequency without loss of accuracy. This arbitrary tuning is efficiently performed by using the step responses of the simple state-space system (13) that requires only the simple operations. In addition, our proposed variable low-pass filters satisfy the property of (12) for any tuning parameter $\eta$. Therefore, by constructing a prototype filter $(A_p, b_p, c_p, d_p)$ in such a manner that it possesses useful properties with respect to quantization effects, our proposed variable low-pass filters show the same useful properties as those of the prototype filter regardless of the value of $\eta$.

IV. NUMERICAL EXAMPLE

This section gives a numerical example to demonstrate the utility of our proposed method from the viewpoint of coefficient quantization effects.

Here we use the following fifth-order transfer function for the prototype filter:

$$
H_p(z) = \frac{\sum_{i=0}^{5} b_i z^{-i}}{1 + \sum_{i=1}^{5} a_i z^{-i}} \tag{22}
$$

where

$$(b_0, b_1, b_2, b_3, b_4, b_5) = (0.0222, -0.0128, 0.0212, 0.0212, -0.0128, 0.0222)$$

$$(a_1, a_2, a_3, a_4, a_5) = (-3.3424, 5.1483, -4.3513, 2.0111, -0.4045). \tag{23}
$$

This transfer function is the elliptic low-pass filter with the peak-to-peak ripple, the minimum stopband attenuation and the passband-edge frequency given as 0.5 dB, 40 dB and 0.25\pi rad, respectively. In our proposed method, we choose the state-space representation of this prototype filter as shown in Table I, which is the balanced realization mentioned in Section II. The controllability/observability Gramians of this state-space system are found to be

$$K_p = \text{diag}(0.9187, 0.7288, 0.4277, 0.1793, 0.0617)$$

$$W_p = K_p. \tag{24}
$$

Figure 1(a) shows the magnitude responses of our proposed variable filter for $\eta = 0, 0.3, 0.6$ and 0.9. Here, all the coefficients of the proposed variable filter are quantized to 9 fractional bits. For comparison purpose, the magnitude responses in the case of the cascaded direct form, which is realized based on [9] and has the same wordlength as our proposed variable filter, is given by Fig. 1(b). Also, the passband magnitude responses of these realizations for $\eta = 0.6$ and $\eta = 0.9$ are respectively shown in Figs. 1(c) and (d).

These results demonstrate that, our proposed variable filter shows very good agreement with the ideal magnitude responses for all $\eta$, whereas the magnitude responses of the cascaded direct form result in severe degradation as $\eta$ becomes closer to unity. The reason for this is clear: as is well-known, narrowband digital filters with direct form structure are very sensitive to quantization effects. On the other hand, as discussed in Sections II and III, an appropriate construction of the controllability/observability Gramians leads to a high-accuracy structure with respect to quantization effects, and our proposed variable filters leave these Gramians to be invariant.
TABLE II
RELATIONSHIP BETWEEN $\eta$ AND $k_c$

<table>
<thead>
<tr>
<th>$\eta$</th>
<th>-0.9</th>
<th>-0.6</th>
<th>-0.3</th>
<th>0.0</th>
<th>0.3</th>
<th>0.6</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_c$</td>
<td>36</td>
<td>12</td>
<td>8</td>
<td>1</td>
<td>7</td>
<td>13</td>
<td>38</td>
</tr>
</tbody>
</table>

for any value of $\eta$. Hence our proposed method shows high tuning accuracy even in the case of narrow bandwidth\(^1\).

Finally we discuss the required number of iterations for computation of $\Psi(k_c)$ by (13). Table II shows the relationship between $\eta$ and $k_c$ for our proposed filter, where $k_c$ is determined as the smallest number that satisfies

$$\frac{|\{(I_N + \eta A_p)^{-1}\}_{i,j} - \Psi_{i,j}(k_c)|}{|\{(I_N + \eta A_p)^{-1}\}_{i,j}|} \leq 0.02$$  \hspace{1cm} (25)

denotes the minimum value that satisfies this constraint. This result shows that larger number of iterations are required as $|\eta|$ approaches to unity, and it remains a future goal to improve this drawback.

V. CONCLUSION

This paper has presented new variable low-pass IIR digital filters with high tuning accuracy. We have derived the proposed variable filters by the state-space-based frequency transformation [8]. Although this transformation requires calculation of an inverse matrix, we have presented an efficient algorithm to obtain this inverse matrix from the step responses of a simple state-space system, which leads to significant simplification of the tuning procedure. In addition, since this algorithm does

---

\(^1\)Although it seems in Fig. 1(d) that the degradation of the magnitude response occurs in our proposed method as well as the cascaded direct form, this degradation can be suppressed by setting the tolerance value on the right-hand side of (25) to be smaller than 0.02.
not rely on the conventional approximation-based methods [6], [7], the proposed variable filters have no limitation on the tuning range. Furthermore, we have shown that the proposed variable filters exhibit high tuning accuracy with respect to quantization effects by appropriately constructing the state-space representation of the prototype low-pass filter.

REFERENCES