CLOSED FORM EXPRESSIONS OF THE MINIMUM $L_2$-SENSITIVITY REALIZATIONS FOR ALL-PASS DIGITAL FILTERS

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Abstract
This paper presents closed form expressions of the minimum $L_2$-sensitivity realizations of all-pass digital filters. We first prove that the $L_2$-sensitivity minimization problem can be solved analytically if the all second-order modes of a digital filter are equal. We focus on the fact that the second-order modes of all-pass digital filters are all unity. As a result, it is clarified that the minimum $L_2$-sensitivity realization of an all-pass digital filter is equal to the balanced realization, of which closed form expression can be derived analytically.

Keywords: State-space digital filters; second-order modes; minimum $L_2$-sensitivity realizations; all-pass digital filters

1 Introduction
On the fixed-point implementation of digital filters, undesirable finite-word-length (FWL) effects arise due to the coefficient truncation and arithmetic roundoff. These FWL effects must be reduced as small as possible because such effects may cause serious degradation of characteristic of digital filters. These undesirable effects can be reduced by synthesizing filters which have low-coefficient quantization error, low-roundoff noise and no limit cycles.

$L_2$-sensitivity is one of the evaluation functions which evaluate the coefficient quantization effects of state-space digital filters. Recently, the $L_2$-sensitivity minimization problems without $L_2$-scaling constraints [1]-[3] and subject to $L_2$-scaling constraints [4]-[6] have been widely investigated. For $L_2$-sensitivity minimization problem without $L_2$-scaling constraints, Yan et al. [1] and Hinamoto et al. [2] proposed solutions using iterative calculations. For $L_2$-sensitivity minimization problem subject to $L_2$-scaling constraints, iterative algorithms using quasi-Newton algorithm [4] and Lagrange multipliers [5] were proposed. However, these solutions cannot give closed form expressions of the minimum $L_2$-sensitivity realizations since they try to solve nonlinear equations by iterative calculations.

On the other hand, our group proposed closed form expressions of the minimum $L_2$-sensitivity realizations without $L_2$-scaling constraints in Ref. [3] and subject to $L_2$-scaling constraints in Ref. [6], respectively. These solutions can obtain the minimum $L_2$-sensitivity realization analytically, that is, without iterative calculations. However, they can be applicable to only second-order digital filters. To the best of our knowledge, there have been no attempts to derive analytical solutions to these problems for high-order digital filters.

In this paper, we propose closed form expressions of the minimum $L_2$-sensitivity realizations for high-order all-pass digital filters. The synthesis procedure does not require iterative calculations. We prove that the $L_2$-sensitivity minimization problem can be solved analytically if the all second-order modes of a digital filter are equal. It is clarified that the minimum $L_2$-sensitivity realization is equal to the balanced realization for all-pass digital filters, all of which second-order modes are equal.

2 Preliminaries
2.1 State-Space digital filters
It is convenient to introduce the state-space model of digital filters for the mathematical analysis. For a given $N$th-order transfer function $H(z)$, a state-space digital filter can be described by the following state-space equations:

$$x(n+1) = Ax(n) + bu(n)$$
$$y(n) = cx(n) + du(n)$$

where $x(n) \in R^{nx1}$ is a state-vector, $u(n) \in R$ is a scalar input, $y(n) \in R$ is a scalar output, and $A \in R^{nxn}, b \in R^{nx1}, c \in R^{1xn}, d \in R$ are coefficient matrices. The block diagram of the state-space digital filter $(A,b,c,d)$ is shown in Fig. 1. The transfer function $H(z)$ is described in terms of the coefficient matrices $(A,b,c,d)$ as

$$H(z) = c(zI - A)^{-1}b + d.$$
2.2 \( L_2 \)-Sensitivity

The \( L_2 \)-sensitivity is one of the measurements which evaluate coefficient quantization effects of digital filters. The \( L_2 \)-sensitivity of the filter \( H(z) \) with respect to the realization \((A,b,c,d)\) is defined by using the general controllability Gramian \( K_i \) and the general observability Gramian \( W_i \) such as [2]

\[
S(A,b,c) = \frac{\partial H(z)}{\partial A} + \frac{\partial H(z)}{\partial b} + \frac{\partial H(z)}{\partial c}.
\]

\[
= tr(W_i)tr(K_i) + tr(W_i) + tr(K_i) + 2 \sum_{i=0}^{2} tr(W_i)tr(K_i).
\]

The general Gramians \( K_i \) and \( W_i \) are defined as solutions to the Lyapunov equations expressed as

\[
K_i = AK_i^T + \frac{1}{2}(A^Tb^T + bb^T(A^T)^T)\]

\[
W_i = A^TW_iA + \frac{1}{2}(c^Ta^T + (A^T)c^Tc)
\]

for \( i=0,1,2,\ldots \), respectively. By simple mathematical manipulation, we derive the novel expressions of general Gramians from Eqs. (5) and (6) as follows:

\[
K_i = \frac{1}{2}(A^TK_i + K_i(A^T)^T)
\]

\[
W_i = \frac{1}{2}(W_iA^T + (A^T)W_i^T)
\]

The controllability Gramian \( K_0 \) and the observability Gramian \( W_0 \) are obtained by letting \( i=0 \) in Eqs. (5) and (6) as follows:

\[
K_0 = AK_0^T + bb^T
\]

\[
W_0 = A^TW_0A + c^Tc
\]

The controllability Gramian \( K_0 \) and the observability Gramian \( W_0 \) are positive definite symmetric, and the eigenvalues \( \theta_i \) \((i=1,\ldots,N)\) of the matrix product \( K_0W_0 \) are all positive. The square roots of the eigenvalues \( \theta_i \) are defined as second-order modes of the filter [7]. In the field of digital signal processing, the controllability and observability Gramian are also called the covariance and noise matrices of the filter \((A,b,c,d)\), respectively [8], [9].

2.3 Coordinate transformation

Let \( T \) be a nonsingular \( N \times N \) real matrix. If a coordinate transformation defined by

\[
\bar{x}(n) = T^{-1}x(n)
\]

is applied to a filter realization \((A,b,c,d)\), we obtain a new realization which has the following coefficient matrices

\[
(\bar{A}, \bar{b}, \bar{c}, \bar{d}) = (T^{-1}AT, T^{-1}b, cT, d)
\]

and the following general Gramians

\[
\bar{K}_i = T^{-1}K_i T^{-T}
\]

\[
\bar{W}_i = T^{T}W_i T
\]

respectively. Letting \( i=0 \) in Eqs. (13) and (14) yields

\[
\bar{K}_0 = T^{-1}K_0T^{-T}
\]

\[
\bar{W}_0 = T^{T}W_0 T
\]

From Eqs. (15) and (16), we have

\[
\bar{K}_0\bar{W}_0 = T^{-1}K_0T^{-T} \bar{W}_0 T
\]

which shows that \( \bar{K}_0\bar{W}_0 \) has the same eigenvalues of \( K_0W_0 \). Thus, the second order modes, which are the square roots of the eigenvalues of \( K_0W_0 \), are invariant under coordinate transformation. The \( L_2 \)-sensitivity of the transformed filter \((T^{-1}AT, T^{-1}b, cT, d)\) can be expressed in terms of the infinite summation of general Gramians as

\[
S(P) = tr(W_0P)tr(K_0P^{-1}) + tr(W_0P) + tr(K_0P^{-1})
\]

\[
+ 2 \sum_{i=0}^{2} tr(W_0P)tr(K_0P^{-1})
\]

where \( P = TT^T \).

2.4 \( L_2 \)-Sensitivity minimization problem

The \( L_2 \)-sensitivity minimization problem is formulated as follows:

For an initial digital filter \((A,b,c,d)\) with a given transfer function \( H(z) \), minimize the \( L_2 \)-sensitivity \( S(P) \) w.r.t. \( P \) where \( P \) is an arbitrary positive definite symmetric matrix.

In general, the balanced realization \((A_0,b_0,c_0,d_0)\) is adopted as the initial realization. The balanced realization \((A_0,b_0,c_0,d_0)\) is a filter structure of which controllability and observability are balanced. For the balanced realization, the controllability and observability Gramians are given by

\[
K_0 = W_0 = \Theta, \quad \Theta = \text{diag}(\theta_0, \ldots, \theta_N)
\]

and the general controllability and observability Gramians are given by

\[
K_i = \frac{1}{2}(A_i^T\Theta + \Theta(A_i^T)^T)
\]

\[
W_i = \frac{1}{2}(\Theta A_i^T + (A_i^T)^T\Theta)
\]

For a given balanced realization \((A_0,b_0,c_0,d_0)\), the minimum \( L_2 \)-sensitivity realization \((A_{opt},b_{opt},c_{opt},d_{opt})\) is obtained by the coordinate transformation matrix \( T_{opt} \) as follows:

\[
\begin{bmatrix}
A_{opt} \\
B_{opt} \\
C_{opt} \\
D_{opt}
\end{bmatrix} =
\begin{bmatrix}
T_{opt}^{-1}A_{opt}T_{opt} \\
T_{opt}^{-1}B_{opt} \\
C_{opt}T_{opt} \\
D_{opt}
\end{bmatrix}
\]

The \( L_2 \)-sensitivity \( S(P) \) has the unique global minimum, which is achieved by \( P_{opt} \) satisfying

\[
\frac{\partial S(P)}{\partial P} = 0
\]
where \( P_{\text{opt}} = T_{\text{opt}} T_{\text{opt}}^T \). The derivative \( \partial S(P) / \partial P \) is given by differentiating \( S(P) \) in (18) with respect to \( P \) as follows:

\[
\frac{\partial S(P)}{\partial P} = \left( 1 + tr(K_i P^{-1}) \right) K_i + 2 \sum_{i=1}^N tr(K_i P^{-1}) W_i - P^{-1} \left( 1 + tr(W_i P) K_i \right) P.
\]

(24)

The algorithm proposed by Yan et al. [1] and the algorithm proposed by Hinamoto et al. [2] derive the matrix \( P_{\text{opt}} \) by iterative calculations for arbitrary order digital filters. The algorithm our group proposed in [3] derives the matrix \( P_{\text{opt}} \) analytically for second-order digital filters. However, there have been no attempts to derive analytical solutions to this problem for high-order digital filters.

3 Closed form expressions of the minimum \( L_2 \)-Sensitivity realizations of All-Pass digital filters

This section proposes closed form expressions of the minimum \( L_2 \)-sensitivity realizations for all-pass digital filters. We propose a theorem on the analytical synthesis of the minimum \( L_2 \)-sensitivity realizations of state-space digital filters all of which second-order modes are equal. We reveal that the \( L_2 \)-sensitivity minimization problem can be solved analytically for high-order all-pass digital filters.

3.1 A closed form expression of the minimum \( L_2 \)-Sensitivity realizations

We have discovered that there exist some digital filters whose minimum \( L_2 \)-sensitivity realization is equal to the balanced realization. Such digital filters satisfy a sufficient condition summarized in the following theorem:

**Theorem 1**

If the all second order modes \( \theta_i (i = 1, \ldots, N) \) of a digital filter \( H(z) \) are equal,

\[
\begin{align*}
(A_{i\text{opt}}, b_{i\text{opt}}, c_{i\text{opt}}, d_{i\text{opt}}) &= (A_0, b_0, c_0, d_0) \quad (25)
\end{align*}
\]

that is, the minimum \( L_2 \)-sensitivity realization without \( L_2 \)-sensitivity constraints is equal to the balanced realization.

**Proof:**

The general Gramians of the balanced realization are given by Eqs. (20) and (21), respectively. If the all second order modes \( \theta_i (i = 1, \ldots, N) \) satisfies

\[
\theta_i = \theta (i = 1, \ldots, N) \quad (26)
\]

the controllability and observability Gramians are expressed as

\[
K_{i\text{opt}}^{(b)} = W_i^{(b)} = \text{diag} (\theta, \ldots, \theta) = \theta. \quad (27)
\]

Substituting Eq. (27) into Eqs. (20) and (21), the general Gramians are given by

\[
K_i^{(b)} = W_i^{(b)} = \frac{1}{2} \theta (A_i^t + (A_i^t)^t) \quad (28)
\]

We can express the general Gramians as \( K_i^{(b)} = W_i^{(b)} = \Theta_i \) which is defined by

\[
\Theta_i = \frac{1}{2} \theta (A_i^t + (A_i^t)^t) \quad (i = 0, 1, \ldots). \quad (29)
\]

Substituting \( K_i^{(b)} = W_i^{(b)} = \Theta_i \) into Eq. (24) yields

\[
\frac{\partial S(P)}{\partial P} = \left( 1 + tr(\Theta_i P^{-1}) \right) \Theta_i + 2 \sum_{i=1}^N tr(\Theta_i P^{-1}) \Theta_i
\]

\[- P^{-1} \left( 1 + tr(W_i P) \Theta_i \right) P^{-1} \quad (30)
\]

\[
= G(P) - P^{-1} G(P) P^{-1}
\]

where

\[
G(P) = \left( 1 + tr(\Theta_i P^{-1}) \right) \Theta_i + 2 \sum_{i=1}^N tr(\Theta_i P^{-1}) \Theta_i
\]

From Eq. (30), it is obvious that

\[
\frac{\partial S(P)}{\partial P} \bigg|_{P=I} = 0 \quad (31)
\]

which means that the minimum \( L_2 \)-sensitivity realization without \( L_2 \)-sensitivity constraints can be synthesized without any coordinate transformation to the balanced realization, that is, the initial realization. Therefore, it is proved that the minimum \( L_2 \)-sensitivity without \( L_2 \)-sensitivity constraints is equal to the balanced realization.

3.2 A closed form expression of the \( L_2 \)-Sensitivity

It is remarkable that the minimum \( L_2 \)-sensitivity \( S_{\text{min}} \) can be expressed in closed form. We obtain the minimum \( L_2 \)-sensitivity \( S_{\text{min}} \) by substituting \( P=I \) into Eq. (18) as follows:

\[
S_{\text{min}} = S(I) = tr(W_0) tr(K_0) + tr(W_0) tr(K_0) + 2 \sum_{i=1}^N tr(W_i) tr(K_i)
\]

(32)

Substituting Eqs. (27) and (28) into Eq. (32) yields

\[
S_{\text{min}} = N^2 \theta^2 + 2 N \theta + 2 \sum_{i=1}^N (A_i^t)^2. \quad (33)
\]

The third term of the right-hand of Eq. (33) converges to a finite value by exploiting the following relation:

\[
tr(A_i^t) = \sum_{p=1}^N \lambda_p \quad (34)
\]

where \( \lambda_p (p = 1, \ldots, N) \) are eigenvalues of \( A_0 \), which are equal to the poles of the digital filter \( H_{\text{zpd}}(z) \). Using the above relation, the minimum \( L_2 \)-sensitivity \( S_{\text{min}} \) are expressed without infinite summations as follows:

\[
S_{\text{min}} = N^2 \theta^2 + 2 N \theta + 2 \sum_{p=1}^N \lambda_p \sum_{i=1}^N \lambda_p \lambda_i \quad (35)
\]

3.3 All-pass digital filters

All-pass digital filters have constant gain at all
It should be noted from the above equation that the balanced realization \((A_{b0},b_{0},c_{0},d_{0})\) is equal to the input normal realization since \((K_0^{(b)})_i=1\) for \(i=1,\cdots,N\). From Theorem 1, the minimum \(L_2\)-sensitivity realization \((A_{opt},b_{opt},c_{opt},d_{opt})\) is equal to the balanced realization \((A_{b0},b_{0},c_{0},d_{0})\). Furthermore, it is remarkable that the balanced realization \((A_{b0},b_{0},c_{0},d_{0})\) satisfies the \(L_2\)-scaling constraints since \((K_0^{(b)})_i=1\) for \(i=1,\cdots,N\). Therefore, the following four filter structures are equal for the all-pass digital filters:

- the balanced realization
- the input normal realization
- the minimum \(L_2\)-sensitivity realization without \(L_2\)-scaling constraints
- the minimum \(L_2\)-sensitivity realization subject to \(L_2\)-scaling constraints.

The relationship among these four filter structures are shown in Fig. 2. Furthermore, the closed form expression of the minimum \(L_2\)-sensitivity \(S_{min}\) can be rewritten by substituting \(\theta_j=1\) in Eq. (35) as

\[
S_{min} = N^2 + 2N + 2N^{N-1} \sum_{j=1}^{N} \frac{\lambda_j^2}{1-\lambda_j^2}
\]  

(39)

![Figure 2. Relationship among four filter structures: Balanced realization, input normal realization, minimum \(L_2\)-sensitivity realization without \(L_2\)-scaling constraints, minimum \(L_2\)-sensitivity realization subject to \(L_2\)-scaling constraints.](image)

### 4 A numerical example

This section gives a numerical example to illustrate the validity of the proposed theorem. Consider a fourth-order all-pass digital filter \(H_{AP}(z)\) given by

\[
H_{AP}(z) = \frac{0.1116 - 0.6948z^{-1} + 1.7233z^{-2} - 2.0456z^{-3} + z^{-4}}{1 - 2.0456z^{-1} + 1.7233z^{-2} - 0.6948z^{-3} + 0.1116z^{-4}}
\]

(40)

of which poles \(\lambda_p (p=1,2,3,4)\) are given by

\[
\lambda_1 = 0.5029 + j0.3765, \quad \lambda_2 = 0.5029 - j0.3765, \quad \lambda_3 = 0.5199 + j0.1119, \quad \lambda_4 = 0.5199 - j0.1119
\]

and of which frequency response and zero-pole plot are shown in Fig. 3. The second-order modes \(\theta_i^{k(b)} (i=1,2,3,4)\) of the all-pass digital filter \(H_{AP}(z)\) are given by

\[
(\theta_1^{k(b)}, \theta_2^{k(b)}, \theta_3^{k(b)}, \theta_4^{k(b)}) = (1,1,1,1).
\]

![Figure 3. Frequency response and zero-pole plot of digital filter \(H_{AP}(z)\).](image)
and controllability Gramian $K_0^{(b)}$ and observability Gramians $W_0^{(b)}$ are calculated as
$$K_0^{(b)} = W_0^{(b)} = (1,1,1). \quad (44)$$

We have to note that the balanced realization $(A_d, b_d, c_d, d_d)$ is equal to the input normal realization $(A,b,c,d)$ since the controllability Gramian $K_0^{(b)}$ is the identity matrix. The minimum $L_2$-sensitivity realization without $L_2$-scaling constraints $(A_{opt}, b_{opt}, c_{opt}, d_{opt})$ is obtained by the algorithm in [2], and the minimum $L_2$-sensitivity realization subject to $L_2$-scaling constraints $(A_{opt}, b_{opt}, c_{opt}, d_{opt})$ is obtained by the algorithm in [5]. Both of them are calculated to be exactly same as the balanced realization $(A_d, b_d, c_d, d_d)$ such as
$$\begin{bmatrix} A_d & b_d \\ c_d & d_d \end{bmatrix} \begin{bmatrix} 0.7525 & -0.1738 & -0.0157 & -0.0015 \\ 0.6552 & 0.2703 & -0.0486 & -0.0157 \\ -0.0370 & 0.8987 & 0.2703 & -0.1738 \\ 0.0398 & -0.0370 & 0.6552 & 0.7525 \\ -0.0393 & 0.2961 & -0.7036 & 0.6351 \end{bmatrix} \begin{bmatrix} 0.6351 \\ -0.7036 \\ 0.6350 \\ 0.6351 \end{bmatrix} \quad (45)$$

respectively. Therefore, for all-pass digital filters, it is numerically confirmed that the following four filter realizations are equal: the balanced realization, the input normal realization, the minimum $L_2$-sensitivity realization without $L_2$-scaling constraints, and the minimum $L_2$-sensitivity realization subject to $L_2$-scaling constraints. The minimum $L_2$-sensitivity $S_{min}$ is calculated as
$$S_{min} = 33.5791 \quad (47)$$
by substituting $N=4$ and Eq. (41) into Eq. (39).

5 Conclusions

This paper presents closed form expressions of the minimum $L_2$-sensitivity realizations for high-order all-pass digital filters. We have clarified the $L_2$-sensitivity minimization problem can be solved analytically when the all second-order modes of the digital filter are equal. In particular, for all-pass digital filters, the following four filter realizations are equal and synthesized analytically: the balanced realization, the input normal realization, the minimum $L_2$-sensitivity realization without $L_2$-scaling constraints, and the minimum $L_2$-sensitivity realization subject to $L_2$-scaling constraints.

References