Design of FIR Digital Filters Without Transition-Band Fluctuations

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Abstract

This paper presents a new method for the design of finite-impulse-response (FIR) digital filters without transition-band fluctuations. We derive a fluctuation-free condition from gradient constraints on the magnitude of transition-band responses. Incorporating the condition, we formulate a new peak-constrained least-squares (PCLS) design of FIR filters as an iterative quadratic programming (QP) problem. Each iteration of the algorithm uses a multiple-exchange technique for giving a QP problem with a small number of constraints. The new algorithm obtains PCLS filters free from transition-band fluctuations efficiently. Design examples are included to illustrate the proposed algorithm.

1. Introduction

The design of finite-impulse-response (FIR) digital filters is generally carried out using the Chebyshev approximation methods [1, 2], the least-squares methods [3, 4], or the peak-constrained least-squares (PCLS) methods [5, 6, 7]. Most of conventional algorithms for the design of FIR filters have a region called transition band between a passband and a stopband. There is no control of the approximation error over the transition band, and it is also called a “don’t care” region. The use of transition bands gives a filter which does not suffer from the well-known Gibbs’ phenomenon and also makes the approximation problem easier. However, the transition bands sometimes exhibit large peaks or undesirable anomalies, especially when there is a big variation in the widths of the transition bands [8] or, in our experience, when the desired group delays are less than about a half the group delay of a linear-phase filter with the same order. We call these phenomena “fluctuations” in the rest of this paper.

In this paper, we first derive a fluctuation-free condition on transition bands. The condition is necessary and sufficient for guaranteeing that there will be no transition-band fluctuations. Next, incorporating the fluctuation-free condition, we formulate a new PCLS design of FIR filters with arbitrary phases as an iterative quadratic programming (QP) problem. Each iteration of the proposed algorithm uses a multiple-exchange technique [7] for giving a QP problem with a small number of constraints, which leads to a great saving of the computational time. Design examples are included to illustrate the proposed algorithm.

2. Condition for Fluctuation-Free Transition Bands

Figure 1 shows the magnitude response of a 36th-order lowpass FIR filter with a desired group delay of six samples. The filter was obtained using the algorithm in [7] with a transition band. This figure shows that there is a fluctuation in the transition band, which gives a filter with a considerably different response than expected. Therefore, when transition bands are used for designing filters, the magnitude responses should always be cared in the transition bands.

Let \( H(e^{j\omega}) \) be the frequency response of an FIR digital filter. The magnitude response \( |H(e^{j\omega})| \) is generally expected to be monotonic in the transition bands. By definition, \( |H(e^{j\omega})| \) is said to be monotonic in the range of \( \Omega \) if and only if either

\[
\frac{\partial |H(e^{j\omega})|}{\partial \omega} \leq 0 \quad \text{or} \quad \frac{\partial |H(e^{j\omega})|}{\partial \omega} \geq 0, \quad \omega \in \Omega.
\]

Note that the former is a condition for monotonically decreasing and the latter is a condition for monotonically increasing. Therefore, \( |H(e^{j\omega})| \) is monotonic in the \( m \)-th transition band.
T_m if and only if
\[
\text{sgn} \left[ \omega_{s,m} - \omega_{p,m} \right] \frac{\partial |H(e^{j\omega})|}{\partial \omega} \leq 0, \quad \omega \in T_m \quad (2)
\]
where \(\text{sgn}[-]\) denotes the signum function of \([-\cdot]\), and \(\omega_{p,m}\) and \(\omega_{s,m}\) denote the passband and stopband edge frequencies on both sides of \(T_m\), respectively. Clearly, (2) is a necessary and sufficient condition for guaranteeing that there will be no transition-band fluctuations.

3. Complex PCLS Design of FIR Filters

Digital filter design based on the peak-constrained least-squares (PCLS) optimality criterion has recently attracted much attention. The PCLS optimality criterion is a generalization of both the least-squares and minimax optimality criteria. It has been shown in [5, 6] that filters optimum in the PCLS sense can efficiently satisfy simultaneous specifications on the peak error and the total squared error, and also shown that a very large reduction in the peak error can be obtained at the expense of a very small increase in the total squared error, and vice versa. In this section, we present a new PCLS design of FIR filters with arbitrary phases by incorporating the fluctuation-free condition.

The frequency response of an \(N\)-th order FIR digital filter is given by the discrete-time Fourier transform of its impulse response \(h(n)\):

\[
H(e^{j\omega}) = \sum_{n=0}^{N} h(n)e^{-j\omega n} = h^T e(\omega) \quad (3)
\]

where

\[
\begin{align*}
\begin{bmatrix}
\begin{array}{c}
\omega
\end{array}
\end{bmatrix}
\end{align*}
\]

In the new PCLS design, we seek a set of filter coefficients \(h\) that solves the constrained problem:

minimize \(f(h) = \int_{\omega \in R} W(\omega) \left| H(e^{j\omega}, h) - D(e^{j\omega}) \right|^2 d\omega \quad (4a)\)

subject to \(\left| H(e^{j\omega}, h) - D(e^{j\omega}) \right| \leq \delta(\omega), \quad \omega \in R \quad (4b)\)

\[
\text{sgn} \left[ \omega_{s,m} - \omega_{p,m} \right] \frac{\partial |H(e^{j\omega}, h)|}{\partial \omega} \leq 0, \quad \omega \in T_m \quad (4c)
\]

where \(R\) is the region \(0 \leq \omega \leq \pi\) excluding the transition bands, \(W(\omega)\) is a non-negative error weighting function, \(\delta(\omega)\) is a constraint function, and \(D(e^{j\omega})\) denotes the desired frequency response with a magnitude response \(M_D(\omega)\) and a phase response \(\phi_D(\omega)\), that is,

\[
D(e^{j\omega}) = M_D(\omega)e^{j\phi_D(\omega)}, \quad \omega \in R. \quad (5)
\]

The weighted-least-squares (WLS) error criterion \(f(h)\) in (4a) can be strictly expressed as a quadratic function with respect to the filter coefficients \(h\):

\[
f(h) = h^T P h - 2h^T q + d \quad (6)
\]

where

\[
\begin{align*}
P &= \int_{\omega \in R} W(\omega) \Re \left[ e(\omega)e^{H(\omega)} \right] d\omega \quad (7) \\
q &= \int_{\omega \in R} W(\omega) \Re \left[ D^*(e^{j\omega})e(\omega) \right] d\omega \quad (8) \\
d &= \int_{\omega \in R} W(\omega) \Re |D(e^{j\omega})|^2 d\omega \quad (9)
\end{align*}
\]

and \(\Re[-]\) denotes the real part of \([-\cdot]\).

It is obvious that both (4b) and (4c) are sets of nonlinear inequalities. Therefore, the PCLS design is essentially a nonlinearly-constrained optimization problem which is difficult to solve. In order to simplify the problem (4), we adopt an iterative procedure [7] by linearly-approximating the sets of nonlinear constraints.

3.1. Linear Approximation of Peak Constraints

For convenience, define the complex error function as \(E(\omega, h) = H(e^{j\omega}, h) - D(e^{j\omega})\) and its argument as \(\phi_E(\omega) = \arg \{E(\omega, h)\}\). In the \(k\)-th iteration of the solution process,

\[
\begin{align*}
\left| E(\omega, h^{(k)}) \right| &\approx \Re \left[ E(\omega, h^{(k)})e^{j\phi_E^{(k-1)}(\omega)} \right] \\
&= a_E(\omega, h^{(k)})ei^{(k)} - \beta(\omega, h^{(k)}) \quad (10)
\end{align*}
\]

where \(a_E(\omega, h^{(k)})\) is a row vector with the \(n\)-th element being \(\cos(\omega n + \phi_E^{(k)}(\omega))\), \(n = 0, \ldots, N\) and

\[
\beta(\omega, h^{(k)}) = M_D(\omega) \cos(\phi_D(\omega) - \phi_E^{(k)}(\omega)) \quad (11)
\]

Hence, the linear approximation of the peak constraints in (4b) is given by

\[
a_E(\omega, h^{(k)})h^{(k)} \leq b(\omega, h^{(k)}), \quad \omega \in R \quad (12)
\]

where

\[
b(\omega, h^{(k)}) = \delta(\omega) + \beta(\omega, h^{(k)}) \quad (13)
\]

3.2. Linear Approximation of Gradient Constraints

Define the argument and group delay of the filter \(H(e^{j\omega}, h^{(k)})\) as

\[
\phi_H^{(k)}(\omega) = \arg \left\{ H(e^{j\omega}, h^{(k)}) \right\} \quad (14)
\]

\[
\tau_H^{(k)}(\omega) = -\frac{\partial \phi_H^{(k)}(\omega)}{\partial \omega} \quad (15)
\]
respectively. In the same manner as in (10),
\[ H(e^{j\omega}, h^{(k)}) \approx \mathbb{R} \left[ H(e^{j\omega}, h^{(k)})e^{-j\phi^{(k-1)}(\omega)} \right] = a_H(\omega, h^{(k)}) \]
where \( a_H(\omega, h^{(k)}) \) is a row vector with the \( n \)-th element being \( \cos(\omega n + \phi^{(k)}(\omega)) \), \( n = 0, \ldots, N \). Hence,
\[ \frac{\partial}{\partial \omega} H(e^{j\omega}, h^{(k)}) \approx \left( \frac{\partial a_H(\omega, h^{(k-1)})}{\partial \omega} \right) h^{(k)} = g(\omega, h^{(k-1)})h^{(k)} \]
(17)
where \( g(\omega, h^{(k)}) \) is a row vector with the \( n \)-th element being
\[ \frac{\partial \cos(\omega n + \phi^{(k)}(\omega))}{\partial \omega} = \left( \phi^{(k)}(\omega) - n \right) \sin(\omega n + \phi^{(k)}(\omega)), \]
(18)
\( n = 0, \ldots, N \). Using (17), the linear approximation of the gradient constraints in (4c) is given by
\[ \text{sgn} \left[ \omega_{s,m} - \omega_{p,m} \right] g(\omega, h^{(k)})h^{(k)} \leq 0, \quad \omega \in T_m. \]
(19)
Consequently, the problem (4) becomes
\[
\begin{align*}
\text{minimize} & \quad f(h^{(k)}) = h^{(k)T} P h^{(k)} - 2h^{(k)T} q + d \\
\text{subject to} & \quad a_E(\omega, h^{(k-1)})h^{(k)} \leq b(\omega, h^{(k-1)}), \quad \omega \in R \\
& \quad \text{sgn} \left[ \omega_{s,m} - \omega_{p,m} \right] g(\omega, h^{(k)})h^{(k)} \leq 0, \quad \omega \in T_m.
\end{align*}
\]
(20)
(20a)
(20b)
(20c)
This problem is obviously a convex quadratic programming (QP) problem. We can easily solve the QP problem at each iteration using a powerful QP tool, such as quadprog in MATLAB.

4. Algorithm
The new PCLS optimization algorithm requires an initial set of filter coefficients \( h^{(0)} \). A convenient way to obtain the initial coefficient set is to solve the unconstrained WLS optimization problem in (20a). The WLS error criterion \( f(h) \) has the global minimum at \( h_{\text{min}} = P^{-1} q \). We can use the analytic solution as the initial coefficient set \( h^{(0)} \).

Each iteration of the algorithm uses a multiple-exchange technique [7] for reformulating a set of peak constraints (20b) and gradient constraints (20c). The multiple-exchange technique formulates the next set of these constraints by releasing several current constraints and setting up several new constraints. The current constraints to be released are the constraints satisfied without equality. In other words, the current constraints satisfied with equality (called active constraints) are also used in the next iteration. The new constraints to be

**Step 1:** Compute \( P \) in (7) and \( q \) in (8) and then initialize \( k = 0 \) and \( h^{(0)} = P^{-1} q \).

**Step 2:** If (4b) or (4c) are violated over permissible limits, (re)formulate a new set of peak and gradient constraints using the multiple-exchange technique described in Sec. 4; else terminate the iterative process. \( h^{(k)} \) is regarded as the optimum solution.

**Step 3:** Set \( k = k + 1 \). Optimize \( h^{(k)} \) by solving the QP problem in (20) and go to Step 2.

Figure 2: Summary of the new PCLS optimization algorithm.

5. Design Examples
We present two examples illustrating the new PCLS optimization algorithm described above. The algorithm was implemented using MATLAB on a Xeon CPU 2.80 GHz.

**Example 1:** A 36th-order lowpass FIR filter was designed with the following specifications:
\[
D(e^{j\omega}) = \begin{cases} 
1, & \omega \in [0, 0.35\pi] \\
0, & \omega \in [0.5\pi, \pi]
\end{cases}
\] (21)
\[
W(\omega) = \begin{cases} 
1 \times 10^{-3}, & \omega \in [0, 0.35\pi] \\
0.999, & \omega \in [0.5\pi, \pi]
\end{cases}
\] (22)
\[
\delta(\omega) = 10^{-39/20}, \quad \omega \in [0, 0.35\pi] \cup [0.5\pi, \pi].
\] (23)
Note that the passband desired group-delay is less than a half the group delay of a linear-phase filter with the same order. With \( \epsilon_{\text{peak}} = 1 \times 10^{-12} \) and \( \epsilon_{\text{grad}} = 1 \times 10^{-4} \), the design took 20 iterations and 3.460 seconds of CPU time. For comparison, the filter was again designed using the conventional PCLS optimization algorithm in [7]. The results obtained are summarized in Table 1. Figure 3 shows both the resulting frequency responses. It is clear from Figure 3(a) that the new PCLS filter has no fluctuation in the transition band. However, the new PCLS filter has less stopband attenuation than the conventional PCLS filter does as you can see in Figure 3(b). This means that the elimination of the transition-band fluctuation was achieved at the expense of increased total-squared-error in the stopband. In fact, the passband-to-stopband energy ratio (PSR) [5] of the new PCLS filter was less than the PSR of the conventional PCLS filter as in Table 1.

**Example 2:** A 60th-order bandpass FIR filter was designed
with the following specifications:

\[
D(e^{j\omega}) = \begin{cases} 
0, & \omega \in [0.2\pi) \cup [0.74\pi \pi] \\
e^{-j0.5\omega}, & \omega \in [0.4\pi \ 0.68\pi] 
\end{cases} \tag{24}
\]

\[
W(\omega) = \begin{cases} 
1 \times 10^{-3}, & \omega \in [0.4\pi \ 0.68\pi] \\
0.999, & \omega \in [0.2\pi \ 0.74\pi] \tag{25}
\end{cases}
\]

\[
\delta(\omega) = 10^{-34/20}, \quad \omega \in [0.2\pi] \cup [0.4\pi \ 0.68\pi] \cup [0.74\pi \pi] \tag{26}
\]

Note that the passband desired group-delay is half the group delay of a linear-phase filter with the same order. In addition, the first transition band is twice the width of the second transition band: thus there is a big variation in the widths of transition bands. The results obtained are summarized in Table 1. The proposed algorithm efficiently achieved the design as compared with the conventional algorithm. Figure 4 shows the resulting frequency responses.

References


Table 1: Summary of the results of design examples

<table>
<thead>
<tr>
<th>Example 1</th>
<th>Example 2</th>
</tr>
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<tr>
<td>Iterations</td>
<td>Method [ ]</td>
</tr>
<tr>
<td>PSR [dB]</td>
<td>48.5094</td>
</tr>
</tbody>
</table>

Figure 3: Frequency responses of lowpass FIR filters designed in Example 1: (a) Magnitude responses. (b) Log-magnitude responses.

Figure 4: Frequency responses of bandpass FIR filters designed in Example 2: (a) Magnitude responses. (b) Passband group delays.