PAPER

Analysis of Second-Order Modes of Linear Continuous-Time Systems under Positive-Real Transformations

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SUMMARY This paper discusses the behavior of the second-order modes (Hankel singular values) of linear continuous-time systems under variable transformations with positive-real functions. That is, given a transfer function $H(s)$ and its second-order modes, we analyze the second-order modes of transformed systems $H(F(s))$, where $1/F(s)$ is an arbitrary positive-real function. We first discuss the case of lossless positive-real transformations, and show that the second-order modes are invariant under any lossless positive-real transformation. We next consider the case of general positive-real transformations, and reveal that the values of the second-order modes are decreased under any general positive-real transformation. We achieve the derivation of these results by describing the controllability/observability Gramians of transformed systems, with the help of the lossless positive-real lemma, the positive-real lemma, and state-space formulation of transformed systems.

key words: linear continuous-time system, second-order mode (Hankel singular value), lossless positive-real transformation, general positive-real transformation, controllability Gramian, observability Gramian

1. Introduction

The second-order modes are defined as the positive square roots of the eigenvalues of the matrix product of the controllability and observability Gramians of linear dynamical systems. In the literature on control system theory, the second-order modes are also called the Hankel singular values because they are represented as the singular values of the Hankel operator of systems.

In various aspects of linear system theory, the second-order modes play crucial roles. One of the well-known examples is balanced model order reduction [1]–[4], where the second-order modes determine the upper bound of the approximation error between the reduced-order model and the full-order model. Another practically important issue can be seen in the field of analog and digital filter theory, where the second-order modes play crucial roles. One of the well-known examples is balanced model order reduction [1]–[4], where the second-order modes determine the upper bound of the approximation error between the reduced-order model and the full-order model. Another practically important issue can be seen in the field of analog and digital filter theory, where the second-order modes provide the optimal dynamic range of analog filters [5], [6], minimum attainable value of roundoff noise of digital filters [7], [8], and minimum attainable value of statistical coefficient sensitivity of digital filters [9]–[11]. Furthermore, the second-order modes characterize the energy storage of systems [12], [13], which is also known to be the Hilbert-Schmidt norm of the Hankel operator [14].

In addition to the practical and theoretical importance mentioned above, the second-order modes have another interesting property, which was discovered by Mullis and Roberts [12]. They proved that the second-order modes of linear discrete-time systems are invariant under any allpass transformation, i.e. under any lossless bounded-real transformation. Our work investigated this invariance property for 2-D discrete systems [15]. Furthermore, in [16] we analyzed the behavior of the second-order modes under general bounded-real transformations, and revealed that the general transformations decrease the values of the second-order modes. These results will be of fundamental importance in studying various subjects of linear system theory, because the second-order modes are closely related to many issues of linear system theory as stated above.

The purpose of this paper is to establish the linear continuous-time counterpart of the above-mentioned theory—we discuss the behavior of the second-order modes of linear continuous-time systems under positive-real transformations. A part of this work was reported in [17], where we proved that the second-order modes are invariant under any typical frequency transformation, i.e. under any transformation such that the transformation function is an arbitrary 1st-order or 2nd-order lossless positive-real function. In this paper, we discuss the case of lossless positive-real transformations of arbitrary order and we derive the same invariance property as in [17]. This result is parallel to the theory of discrete-time lossless transformations [12]. Furthermore, we also discuss the case of general positive-real transformations and we reveal that any of the general transformations decreases the values of the second-order modes. This result is parallel to the theory of discrete-time general transformations [16].

The organization of this paper is as follows. Section 2 gives preliminaries, where we introduce the second-order modes and positive-real transformations of linear continuous-time systems. Sections 3 and 4 present our main results that establish the theory on the second-order modes of continuous-time systems under positive-real transformations. Section 3 discusses the case of lossless positive-
real transformations, and Sect. 4 deals with general positive-real transformations. Section 5 gives numerical example to demonstrate our main results.

2. Preliminaries

2.1 Second-Order Modes of Linear Continuous-Time Systems

Consider the following state-space equations for an asymptotically stable multi-input/multi-output continuous-time system of order $N$:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

(1)

$$y(t) = Cx(t) + Du(t)$$

(2)

where $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ and $x(t) \in \mathbb{R}^N$ are the input, output and state of the system, and $A \in \mathbb{R}^{N \times N}$, $B \in \mathbb{R}^{N \times m}$, $C \in \mathbb{R}^{p \times N}$ and $D \in \mathbb{R}^{p \times m}$ are constant coefficient matrices. The coefficient matrices and the transfer function $H(s)$ are related as

$$H(s) = D + C(sI_N - A)^{-1}B$$

(3)

where $I_N$ is the $N \times N$ identity matrix. Throughout this paper, the system $(A, B, C, D)$ is assumed to be a minimal realization of $H(s)$, i.e. the system is controllable and observable.

For the system $(A, B, C, D)$, the solutions $K$ and $W$ to the following Lyapunov equations are called the controllability Gramian and the observability Gramian, respectively:

$$AK + KA^T + BB^T = 0$$

(4)

$$A^TW + WA + C^TC = 0.$$ 

(5)

The Gramians $K$ and $W$ are symmetric and positive definite, i.e. $K = K^T > 0$ and $W = W^T > 0$, because the system $(A, B, C, D)$ is assumed to be asymptotically stable, controllable and observable. Then, the eigenvalues $\theta_1^2, \theta_2^2, \ldots, \theta_N^2$ of the matrix product $KW$ are all positive. The positive square roots $\theta_1, \theta_2, \ldots, \theta_N$ of the eigenvalues are called the second-order modes of the system [1], [7].

It should be noted that the Gramians depend on realization of the system, while the second-order modes depend only on the transfer function. In the literature on control system theory, the second-order modes are also called Hankel singular values because the eigenvalues of $KW$ are equal to the singular values of the Hankel operator of $H(s)$.

2.2 Positive-Real Transformations

As stated in Sect. 1, the purpose of this paper is to discuss the second-order modes under positive-real transformations. That is, we will analyze the relationship between the second-order modes of $H(s)$ and those of $H(F(s))$, where $1/F(s)$ is an arbitrary positive-real function. To this end, in this subsection we review the positive-real transformations and their state-space descriptions.

We begin with the definition of the positive-real transformations as follows.

**Definition 1** (Positive-real transformations): For a given transfer function $H(s)$, consider a variable transformation $1/s \leftarrow 1/F(s)$ which results in a new transfer function $H(F(s))$. To guarantee the stability of $H(F(s))$, the function $1/F(s)$ must be positive-real, which means that $1/F(s) + 1/F(s') \geq 0$ holds for $\text{Re}(s) > 0$. Such a variable transformation $1/s \leftarrow 1/F(s)$ is referred to as positive-real transformation. A simple example of positive-real functions is RLC driving-point impedance functions.

The special case of positive-real transformations is lossless positive-real transformations. The lossless transformations are also called frequency transformations, and they appear in the literature on analog filter design [18]. The definition is given as follows.

**Definition 2** (Lossless positive-real transformations): For a given transfer function $H(s)$, a variable transformation $1/s \leftarrow 1/F(s)$ is called lossless positive-real transformation if $1/F(s)$ is lossless positive-real. In this case, $1/F(s)$ satisfies $1/F(s) + 1/F(-s) = 0$ for all $s$ such that $s$ is not a pole of $1/F(s)$. Such functions are represented as the following form:

$$\frac{1}{F(s)} = \frac{A_0}{s} + \sum_{k=1}^{l} \frac{2A_k s}{s^2 + \omega_k^2} + A_\infty s$$

(6)

where $A_0$, $A_k$, and $A_\infty$ are real nonnegative coefficients. Equation (6) is referred to as reactance function, which is well-known as an LC driving-point impedance function.

We next introduce state-space descriptions of positive-real and lossless positive-real functions. Such descriptions are well-known as the positive-real lemma and the lossless positive-real lemma [19].

**Lemma 1** (Positive-real lemma): Let $1/F(s)$ be a proper function (see Remark 1), and let $(\alpha, \beta, \gamma, \delta)$ be a state-space representation of $1/F(s)$. If $1/F(s)$ is positive-real, there exist a positive definite matrix $P$ and a real row vector $l$ such that

$$\alpha^T P + P \alpha = -l^T l$$

(7)

$$P \beta = \gamma^T - \sqrt{2l^T l}.$$ 

(8)

Equations (7) and (8) are called the positive-real lemma\(^1\).

**Lemma 2** (Lossless positive-real lemma): Let $1/F(s)$ and $(\alpha, \beta, \gamma, \delta)$ be given as above. If $1/F(s)$ is lossless positive-real, $\delta = 0$ and $l = 0$ hold, and thus (7), (8) are reduced to

$$\alpha^T P + P \alpha = 0$$

(9)

$$P \beta = \gamma^T.$$ 

(10)

Equations (9) and (10) are called the lossless positive-real lemma.\(^1\)

\(^1\)Since the function $1/F(s)$ is defined as a single-input/single-output function in this paper, we consider the positive-real lemma with respect to single-input/single-output systems.
Remark 1: A rational function \( G(s) = N(s)/D(s) \) is called proper if \( \deg N(s) \leq \deg D(s) \). Especially, \( G(s) \) is called strictly proper if \( \deg N(s) < \deg D(s) \). On the other hand, \( G(s) \) is called improper if \( \deg N(s) > \deg D(s) \).

Before leaving this section, we introduce state-space formulation of \( H(F(s)) \), as stated in the following lemma [12].

Lemma 3: Let \((A, B, C, D)\) be a state-space representation of an \( N \)-th order function \( H(s) \). Also, let \((\alpha, \beta, \gamma, \delta)\) be a state-space representation of an \( M \)-th order function \( 1/F(s) \), and assume that \( 1/F(s) \) is proper. Then, the transformed system \( H(F(s)) \), which is of order \( MN \), is described in state-space form by

\[
H(F(s)) = D + C(\delta I_{MN} - \mathcal{A})^{-1}\mathcal{B}
\]

where the coefficients \((\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})\) are given as

\[
\mathcal{A} = I_N \otimes \alpha + [A(I_N - \delta A)^{-1}] \otimes (\beta \gamma)
\]

\[
\mathcal{B} = [(I_N - \delta A)^{-1}B] \otimes \beta
\]

\[
\mathcal{C} = [C(I_N - \delta A)^{-1}] \otimes \gamma
\]

\[
\mathcal{D} = D + \delta C(I_N - \delta A)^{-1}B.
\]

In (12)–(15), \( \otimes \) denotes the Kronecker product for matrices [20].

Remark 2: In [12], the state-space formulation of transformed systems was discussed for single-input/single-output discrete-time systems. However, it readily follows that the formulation in [12] is also available for multi-input/multi-output systems or continuous-time systems as it is. Therefore, equations (12)–(15) are directly introduced from [12]. Note that the inverse matrix \((I_N - \delta A)^{-1}\) exists because \( \delta \geq 0 \) holds by the positive-realness of \( 1/F(s) \) and \((A, B, C, D)\) is assumed to be asymptotically stable.

3. Second-Order Modes under Lossless Positive-Real Transformations

This section and the next section present our main results, where we reveal the relationship between the second-order modes of \( H(s) \) and those of \( H(F(s)) \). In this section, we discuss the case of lossless positive-real transformations. That is, the function \( 1/F(s) \) is restricted to lossless positive-real.

For analysis purpose, we classify lossless positive-real functions into two categories — strictly proper functions and improper functions\(^1\). On the basis of this classification, we attempt to analyze the second-order modes of \( H(F(s)) \). Throughout this section, the transformation \( 1/s \rightarrow 1/F(s) \) is referred to as strictly proper lossless positive-real transformation if \( 1/F(s) \) is strictly proper and lossless positive-real. Similarly, if \( 1/F(s) \) is improper and lossless positive-real, \( 1/s \rightarrow 1/F(s) \) is referred to as improper lossless positive-real transformation.

First, we discuss the case of strictly proper lossless positive-real transformations. Since the second-order modes are obtained from the controllability and observability Gramians, we need to describe the controllability/observability Gramians of transformed systems in terms of state-space representations of \( H(s) \) and \( 1/F(s) \). The following lemma presents the description.

Lemma 4: Let \( K \) and \( W \) be the controllability and observability Gramians of \((A, B, C, D)\) in \( H(s) \). Also, let \( \mathcal{K} \) and \( \mathcal{W} \) be the controllability and observability Gramians of \((\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})\) in \( H(F(s)) \) given by Lemma 3, where \( 1/F(s) \) is strictly proper and lossless positive-real. Note that, by definition, \( \mathcal{K} \) and \( \mathcal{W} \) are the solutions to the following Lyapunov equations:

\[
\mathcal{A}\mathcal{K} + \mathcal{K}\mathcal{A}^T + 2\mathcal{B}\mathcal{B}^T = 0
\]

\[
\mathcal{A}^T\mathcal{W} + \mathcal{W}\mathcal{A} + C^T C = 0.
\]

Then, \( \mathcal{K} \) and \( \mathcal{W} \) are respectively related to \( K \) and \( W \) as follows:

\[
\mathcal{K} = K \otimes P^{-1}
\]

\[
\mathcal{W} = W \otimes P
\]

where \( P \) is the positive definite matrix that satisfies the lossless positive-real lemma (9), (10).

Proof: The proof of Lemma 4 makes use of the following properties for the Kronecker product [20]:

\[
(U \otimes V)(X \otimes Y) = (UX) \otimes (VY)
\]

\[
(U + V)(X + Y) = U \otimes X + U \otimes Y + V \otimes X + V \otimes Y
\]

\[
(U \otimes V)^{-1} = U^{-1} \otimes V^{-1}
\]

\[
(U \otimes V)^T = U^T \otimes V^T.
\]

Here we give the proof of (19). Substituting \( \mathcal{W} = W \otimes P \) into the left-hand side of (17) and using (12) and (14) in Lemma 3 with \( \delta = 0 \), we have

\[
\mathcal{A}^T(\mathcal{W} \otimes P) + (\mathcal{W} \otimes P)\mathcal{A} + C^T C
\]

\[
=[I_N \otimes \alpha^T + A^T \otimes (\gamma^T \beta^T)](W \otimes P)
\]

\[+(W \otimes P)[I_N \otimes \alpha + A \otimes (\beta \gamma)]
\]

\[+(C^T \otimes \gamma^T)(C \otimes \gamma)
\]

\[= W \otimes (\alpha^T P + P \alpha) + (A^T W) \otimes (\gamma^T \beta^T P)
\]

\[+(WA) \otimes (P \beta \gamma) + (C^T C) \otimes (\gamma^T \gamma).
\]

By the lossless positive-real lemma (9), (10), the right-hand side of the final equality in (24) is simplified to

\[
W \otimes (\alpha^T P + P \alpha) + (A^T W) \otimes (\gamma^T \beta^T P)
\]

\[+(WA) \otimes (P \beta \gamma) + (C^T C) \otimes (\gamma^T \gamma)
\]

\[= 0.
\]

\[^1\text{From (6), we see that lossless positive-real functions do not have the form such that the degrees of the numerator and denominator polynomials are identical. Therefore, a proper but not strictly proper function does not exist in the family of lossless positive-real functions.}

\[^2\text{In this case} \delta = 0 \text{holds for all} 1/F(s), \text{because} 1/F(s) \text{is assumed to be strictly proper.} \]
which shows \( \mathcal{A}^T(W \otimes P) + (W \otimes P)\mathcal{A} + C^T C = 0 \) and completes the proof of (19). The proof of (18) is given in a similar way and thus omitted. \( \square \)

From Lemma 4 easily see \( \mathcal{K}W = (KW) \otimes I_M \), which shows that the eigenvalues of \( \mathcal{K}W \) are the same as those of \( KW \) with multiplicity \( M \). Hence we obtain the following proposition.

**Proposition 1:** Let \( \theta_1 \geq \theta_2 \geq \cdots \geq \theta_N \) be the second-order modes of \( H(s) \). Also, let \( 1/F(s) \) be an \( M \)-th order strictly proper lossless positive-real function, and let \( \theta_1 \geq \theta_2 \geq \cdots \geq \theta_{MN} \) be the second-order modes of \( H(F(s)) \). Then, the following holds:

\[
\bar{\theta}_j = \theta_i \tag{26}
\]

where \( 1 \leq i \leq N \) and \( M(i - 1) + 1 \leq j \leq M_i \). \( \square \)

This proposition shows that the second-order modes of \( H(F(s)) \) are the same as those of \( H(s) \), with multiplicity \( M \). That is, the values of the second-order modes are invariant under any strictly proper lossless positive-real transformation.

Next we turn our attention to the case of improper lossless positive-real transformations. As stated in Remark 1, an improper function means that the degree of the denominator in the function is smaller than that of the numerator. As is well-known, such a transfer function cannot be described in state-space form. Hence, when \( 1/F(s) \) is improper, unfortunately Lemma 3 is not applicable to obtain a state-space representation of \( H(F(s)) \). Therefore, in this case an alternative approach is required to describe \( H(F(s)) \) in state-space form.

Our strategy to be presented here makes use of \( F(s) \) instead of \( 1/F(s) \). Using this approach, we express the transformed systems \( H(F(s)) \) as follows:

\[
H(F(s)) = H(s)|_{s=1/F(s)} = H\left(\frac{1}{F(s)}\right)|_{s=1/F(s)}. \tag{27}
\]

Since \( 1/F(s) \) is improper, \( F(s) \) becomes strictly proper and it can be described in state-space form. In addition, if \( 1/F(s) \) is (lossless) positive-real, so is \( F(s) \). Furthermore, given a transfer function \( H(s) \) and its state-space representation \( (A, B, C, D) \), the transfer function \( H(1/s) \) can be represented in state-space form as \((A^{-1}, -A^{-1}B, CA^{-1}, D - CA^{-1}B)\) \cite{21,22}. These facts enable the state-space formulation of variable transformations with improper functions. The formulation now follows.

**Lemma 5:** Let \( (A, B, C, D) \) be a state-space representation of \( H(s) \). Similarly, let \( 1/F(s) \) be an improper function and let \( (a_1, b_1, c_1, d_1) \) be a state-space representation of \( F(s) \). Note that \( d_1 = 0 \) because \( F(s) \) is strictly proper. Then, \( H(F(s)) \) is described in state-space form by

\[
H(F(s)) = D_1 + C_1(sI_{M_i} - A_1)^{-1}B_1 \tag{28}
\]

where \( M_i \) denotes the order of \( F(s) \) and the coefficients \( (A_1, B_1, C_1, D_1) \) are given as

\[
A_1 = I_N \otimes \alpha_1 + A^{-1} \otimes (\beta_i \gamma_i) \tag{29}
\]

\[
B_1 = (-A^{-1}B) \otimes \beta_i \tag{30}
\]

\[
C_1 = (CA^{-1}) \otimes \gamma_i \tag{31}
\]

\[
D_1 = D - CA^{-1}B. \tag{32}
\]

Lemma 5 and the lossless positive-real lemma lead to the description of the controllability/observability Gramians under improper lossless positive-real transformations. This description is given in the following lemma.

**Lemma 6:** Let \( K \) and \( W \) be the controllability and observability Gramians of \((A, B, C, D)\) in \( H(s) \). Also, let \( \mathcal{K}_I \) and \( \mathcal{W}_I \) be the controllability and observability Gramians of \((\mathcal{A}_I, \mathcal{B}_I, \mathcal{C}_I, \mathcal{D}_I)\) in \( H(F(s)) \) given by Lemma 5. If \( 1/F(s) \) is lossless positive-real, \( \mathcal{K}_I \) and \( \mathcal{W}_I \) are represented as

\[
\mathcal{K}_I = K \otimes P^{-1}_I \tag{33}
\]

\[
\mathcal{W}_I = W \otimes P_I \tag{34}
\]

where \( P_I \) is the positive definite matrix that satisfies the lossless positive-real lemma for \((\alpha_1, \beta_1, \gamma_1)\), i.e.

\[
\alpha_1^T P_I + P_I \alpha_1 = 0 \tag{35}
\]

\[
P_I \beta_1 = \gamma_1^T \tag{36}
\]

**Proof:** Here we give the proof of (34), which is similar to the proof of (19) in Lemma 4. Consider

\[
\mathcal{A}^T(W \otimes P_I) + (W \otimes P_I)\mathcal{A} + C^T C = 0 \tag{37}
\]

Using (35) and (36), we can simplify the right-hand side of the final equality in (37) to

\[
W \otimes (\alpha_1^T P_I + P_I \alpha_1) + (A^{-T}W) \otimes (\gamma_1^T \beta_1^T P_I)
+ (WA^{-1}) \otimes (P_I \beta_1 \gamma_1) + (A^{-T}C^T CA^{-1}) \otimes (\gamma_1^T \gamma_1)
= (A^{-T}W + WA^{-1} + A^{-T}C^T CA^{-1}) \otimes (\gamma_1^T \gamma_1)
= (A^{-T}(A^{-T}W + WA + C^T CA^{-1})) \otimes (\gamma_1^T \gamma_1)
= 0. \tag{38}
\]

This result shows \( \mathcal{A}_I^T(W \otimes P_I) + (W \otimes P_I)\mathcal{A}_I + C^T C = 0 \), which completes the proof of (34). The proof of (33) is given in a similar way and omitted here.

Lemma 6 leads to \( \mathcal{K}_I \mathcal{W}_I = (KW) \otimes I_M \), which yields the same result as in the case of strictly proper transformations. Consequently, we obtain the following proposition.
Proposition 2: Let \( \theta_1 \geq \theta_2 \geq \cdots \geq \theta_N \) be the second-order modes of \( H(s) \). Also, let \( 1/F(s) \) be an improper lossless positive-real function, and let \( \overline{\theta}_1 \geq \overline{\theta}_2 \geq \cdots \geq \overline{\theta}_{M,N} \) be the second-order modes of \( H(F(s)) \). Then, the following holds:
\[
\overline{\theta}_j = \theta_i
\]
where \( 1 \leq i \leq N \) and \( M_i(i-1) + 1 \leq j \leq M_i \). That is, the second-order modes are invariant under any improper lossless positive-real transformation.

From Propositions 1 and 2 we finally present the following theorem, which explicitly states the behavior of the second-order modes under lossless positive-real transformations.

Theorem 1: The second-order modes of linear continuous-time systems are invariant under any lossless positive-real transformation. That is, given a stable transfer function \( H(s) \) and an arbitrary lossless positive-real function \( 1/F(s) \), the values of the second-order modes of \( H(F(s)) \) are the same as those of \( H(s) \).

Remark 3: In our previous work [17], this invariance property is discussed for only a subclass of lossless positive-real transformations—the transformations are restricted to 1st-order or 2nd-order lossless positive-real functions. On the other hand, in this paper we have derived the invariance property for arbitrary lossless positive-real transformations.

The significance of Theorem 1 lies in the fact that, given the second-order modes of \( H(s) \), we can simultaneously know the second-order modes of any other system \( H(F(s)) \) that is generated from \( H(s) \) by a lossless positive-real transformation. This fact will be very useful for analysis of structural properties of linear dynamical systems, which will be also beneficial to practical issues. For example, in balanced model order reduction, Theorem 1 tells us that the upper bound of the approximation error with respect to \( H(F(s)) \) is simply characterized by the upper bound with respect to \( H(s) \). Therefore, once we evaluate the approximation error of a system \( H(s) \), we can immediately evaluate the approximation error of any other system \( H(F(s)) \).

For discrete-time systems, the second-order modes under lossless transformations were discussed in [12], and the same invariance property as in Theorem 1 was derived. Hence our result is parallel to the theory of discrete-time lossless transformations.

4. Second-Order Modes under General Positive-Real Transformations

In this section we discuss the case of general positive-real transformations. As similar to the previous section, we classify the family of general positive-real functions into two categories—proper functions and improper functions.

First, we consider the case where \( 1/F(s) \) is proper. In this case Lemma 3 is applicable to the state-space formulation of \( H(F(s)) \), which leads to analysis of the Gramians under proper general positive-real transformations. The following Lemma shows the result.

Lemma 7: Let \( K \) and \( W \) be the controllability and observability Gramians of \( (A, B, C, D) \) in \( H(s) \). Also, let \( K' \) and \( W' \) be the controllability and observability Gramians of \( (A', B', C', D') \) in \( H(F(s)) \) given by Lemma 3. If \( 1/F(s) \) is proper and positive-real, \( K \) and \( W \) are respectively related to \( K' \) and \( W' \) as follows:
\[
K \leq K \otimes P^{-1}
\]
\[
W \leq W \otimes P
\]
where \( P \) is a positive definite matrix that satisfies the positive-real lemma (7), (8) for \( (\alpha, \beta, \gamma, \delta) \). Equality holds if and only if \( 1/F(s) \) is lossless positive-real.

Proof: We give the proof of (41). Let \( E \) denote \( (I_N - \delta A) \) and consider
\[
A^T(W \otimes P) + (W \otimes P)A + C^T C
\]
\[
= \left[ I_N \otimes \alpha + (E^{-T} A^T) \otimes (\gamma^T \beta^T) \right] (W \otimes P)
\]
\[
+ (W \otimes P) \left[ I_N \otimes \alpha + (AE^{-1}) \otimes (\beta \gamma) \right]
\]
\[
+ \left[ (E^{-T} C^T) \otimes \gamma \right] \left[ (CE^{-1}) \otimes \gamma \right]
\]
\[
= W \otimes (\alpha_1^T + (E^{-T} A^T W + WA E^{-1}) \otimes (\beta_1 \gamma_1))
\]
\[
+ (E^{-T} C^T CE^{-1}) \otimes (\gamma_1 \gamma_2)
\]
\[
= W \otimes (\alpha_2^T) - \sqrt{2\delta}(E^{-T} A^T W \otimes (\gamma_1 \gamma_1))
\]
\[
- \sqrt{2\delta}(WA E^{-1}) \otimes (\gamma_2 \gamma_2)
\]
\[
+ (E^{-T} A^T W + WA E^{-1} + E^{-T} C^T CE^{-1}) \otimes (\gamma_1^T \gamma_1)
\]
\[
= \left[ W \otimes (\alpha_2) + (E^{-T} A^T W) \otimes (\gamma_1 \gamma_1) \right]
\]
\[
+ \sqrt{2\delta}(WA E^{-1}) \otimes (\gamma_2 \gamma_2)
\]
\[
+ 2\delta(E^{-T} A^T W A E^{-1}) \otimes (\gamma_1 \gamma_1)
\]
where the third equality follows from the positive-real lemma, and the final equality follows from the following relationship:
\[
E^{-T} A^T W + WA E^{-1} + E^{-T} C^T CE^{-1}
\]
\[
= E^{-T} (A^T W A + C^T C) E^{-1}
\]
\[
= E^{-T} [A^T W (I_N - \delta A)
\]
\[
+ (I_N - \delta A^T W A + C^T C] E^{-1}
\]
\[
= E^{-T} (A^T W + WA + C^T C - 2\delta A^T W A) E^{-1}
\]
\[
= -2\delta E^{-T} A^T W A E^{-1}
\]
Letting \( R = W^{\frac{1}{2}} \otimes I + \sqrt{2\delta}(W^{\frac{1}{2}} A E^{-1}) \otimes \gamma \), we can rewrite (42) as
\[
A^T(W \otimes P) + (W \otimes P)A + C^T C = -R^T R
\]
Now, from (44) and the Lyapunov equation \( A^T W + WA = -C^T C \), we obtain the following equation:
\[
A^T(W \otimes P - \gamma W) + (W \otimes P - \gamma W)A
\]
\[
= -R^T R
\]
Since $H(F(s))$ is stable and $R^TR \succeq 0$, Eq. (45) implies that the matrix $W \otimes P - W'$ is positive semidefinite. Hence we obtain $W \otimes P \succeq W'$, which completes the proof of (41). The equality of (41) is trivial by the lossless positive-real lemma. The proof of (40) can be achieved in a similar way and omitted here.

Lemma 7 results in an eigenvalue inequality with respect to $\mathcal{K}W$ and $(KW) \otimes I_M$, as shown in the following lemma.

**Lemma 8:** Let $K, W, \mathcal{K}, W$ and $P$ be the positive definite matrices that satisfy Lemma 7. Then, the following eigenvalue inequality holds:

$$
\lambda_k(KW) \leq \lambda_k((KW) \otimes I_M)
$$

where $1 \leq k \leq MN$ and $\lambda_k(X)$ denotes the $k$th-largest eigenvalue of a matrix $X$, i.e. $\lambda_1(X) \geq \lambda_2(X) \geq \cdots \geq \lambda_{MN}(X)$.

**Proof:** For simplicity of notations, let $K' = K \otimes P^{-1}$ and $W' = W \otimes P$. From (40) and (41), we obtain

$$
\begin{align*}
K^{-\frac{1}{2}}K'K^{-\frac{1}{2}} & \leq I_M, \\
K'W'K^{-\frac{1}{2}} & \leq K'W'K^{-\frac{1}{2}},
\end{align*}
$$

which yield the following eigenvalue inequalities for $1 \leq k \leq MN$:

$$
\begin{align*}
\lambda_k(K^{-\frac{1}{2}}K'K^{-\frac{1}{2}}) & \leq 1, \\
\lambda_k(K'W'K^{-\frac{1}{2}}) & \leq \lambda_k(K'W'K^{-\frac{1}{2}}).
\end{align*}
$$

Now, consider

$$
\begin{align*}
\lambda_k(KW) \\
= \lambda_k(K'K^{-\frac{1}{2}}K'K^{-\frac{1}{2}}K'W'K^{-\frac{1}{2}}) \\
= \lambda_k((W'K^{-\frac{1}{2}}K'K^{-\frac{1}{2}})W'K^{-\frac{1}{2}}) \\
= \lambda_k((W'K^{-\frac{1}{2}})K'K^{-\frac{1}{2}}(W'K^{-\frac{1}{2}})^T)
\end{align*}
$$

where the second equality follows from the fact that $\lambda_k(UU) = \lambda_k(VV)$ for $U > 0$ and $V > 0$. Applying the following relationship [23]

$$
\lambda_k(YY^T) \leq \lambda_k(X)\lambda_k(Y^TY), \quad X = X^T
$$

to (51), we obtain

$$
\begin{align*}
\lambda_k(KW) & \leq \lambda_k((W'K^{-\frac{1}{2}})K'K^{-\frac{1}{2}}W'K^{-\frac{1}{2}})
\end{align*}
$$

which completes the proof.

Proposition 3: Let $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_N$ be the second-order modes of $H(s)$. Also, let $1/F(s)$ be a proper positive-real function, and let $\theta_1, \theta_2, \theta_N$ be the second-order modes of $H(F(s))$. Then, the following holds:

$$
\theta_j \leq \theta_i
$$

where $1 \leq i \leq N$ and $M(i-1) + 1 \leq j \leq M_i$. Equality holds for all $i$ and $j$ if and only if $1/F(s)$ is lossless positive-real.

**Proof:** As stated earlier, the eigenvalues of $(KW) \otimes I_M$ are given as the $M$ copies of the eigenvalues of $KW$. Using this fact and Lemma 8, we immediately obtain

$$
\lambda_j(KW) \leq \lambda_i(KW)
$$

where $1 \leq i \leq N$ and $M(i-1) + 1 \leq j \leq M_i$. Hence it follows that $\theta_j \leq \theta_i$. The equality is trivial by Proposition 1.

Proposition 3 shows that the values of the second-order modes are decreased under proper general positive-real transformations. From this result, we see that Proposition 1 is the special case of Proposition 3.

Next, we discuss the case where $1/F(s)$ is improper. Before giving the main result, we derive matrix inequalities with respect to the controllability/observability Gramians in $H(s)$ and $H(F(s))$, as stated in the following lemma.

**Lemma 9:** Let $K$ and $W$ be the controllability and observability Gramians of $(A, B, C, D)$ in $H(s)$. Also, let $\mathcal{K}_I$ and $\mathcal{W}_I$ be the controllability and observability Gramians of $(\mathcal{A}_I, \mathcal{B}_I, \mathcal{C}_I, \mathcal{D}_I)$ in $H(F(s))$ given by Lemma 5. If $1/F(s)$ is positive-real, $\mathcal{K}_I$ and $\mathcal{W}_I$ are respectively related to $K$ and $W$ as

$$
\begin{align*}
\mathcal{K}_I & \leq K \otimes P_I^{-1}, \\
\mathcal{W}_I & \leq W \otimes P_I
\end{align*}
$$

where $P_I$ is a positive definite matrix that satisfies the positive-real lemma for $(\alpha_I, \beta_I, \gamma_I, 0)$, i.e.

$$
\begin{align*}
\alpha_I^2 P_I + P_I \alpha_I &= -I_I^T I_I, \\
\beta_I \alpha_I &= \gamma_I^T I_I
\end{align*}
$$

with some $I_I$. Equality holds if and only if $1/F(s)$ is lossless positive-real.

**Proof:** We give the proof of (57). Consider

$$
\begin{align*}
\mathcal{A}_I^T (W \otimes P_I) + (W \otimes P_I) \mathcal{A}_I + C_I \mathcal{K}_I \\
= [I_N \otimes \alpha_I^2 + A^{-T} \otimes (\gamma_I^T \beta_I)] (W \otimes P_I) \\
+ (W \otimes P_I) [I_N \otimes \alpha_I + A^{-1} \otimes (\beta_I \gamma_I)] \\
+ [(A^{-T} C_I^T) \otimes (C A^{-1}) \otimes \gamma_I] \\
= W \otimes \alpha_I^2 P_I + P_I \alpha_I + (A^{-T} W) \otimes (\gamma_I^T \beta_I P_I) \\
+ (A W^{-1}) \otimes (P_I \beta_I \gamma_I) + (A^{-T} C_I^T C A^{-1}) \otimes (\gamma_I^T \gamma_I) \\
= W \otimes (-I_I^T I_I) \\
+ [A^{-T} (A^T W + W A + C^T C) A^{-1}] \otimes (\gamma_I^T \gamma_I) \\
= -W \otimes (I_I^T I_I) \\
= -[W^\perp \otimes I_I^T] [W^\perp \otimes I_I]
\end{align*}
$$

(60)
where the third equality follows from (58), (59). Letting \( S = W^2 \otimes I_1 \), we have

\[
\mathcal{A}_f^T (W \otimes P_1) + (W \otimes P_1) \mathcal{A}_f + C_1^T C_f = -S^T S. \tag{61}
\]

Now, from (61) and the fact that \( \mathcal{A}_f^T W_f + W_f \mathcal{A}_f + C_1^T C_f = 0 \), we obtain

\[
\mathcal{A}_f^T (W \otimes P_1 - W_f) + (W \otimes P_1 - W_f) \mathcal{A}_f = -S^T S. \tag{62}
\]

Since \( S^T S \geq 0 \), it readily follows that the matrix \( W \otimes P_1 - W_f \) is positive semidefinite. Hence we obtain (57). The equality of (57) follows from the lossless positive-real lemma. The proof of (56) is omitted here for brevity.

From Lemma 9 we know \( \lambda_k(K_i^T W_f) \leq \lambda_k((KW)I_{M_1}) \) for \( 1 \leq k \leq M_1N \), which is the same result as in the case of proper general transformations. Now, we provide the following proposition.

**Proposition 4:** Let \( \theta_1 \geq \theta_2 \geq \cdots \geq \theta_N \) be the second-order modes of \( H(s) \). Also, let \( 1/F(s) \) be an improper positive-real function, and let \( \theta_1 \geq \theta_2 \geq \cdots \geq \theta_{M_1N} \) be the second-order modes of \( H(F(s)) \). Then, the following holds:

\[
\theta_j \leq \theta_i \tag{63}
\]

where \( 1 \leq i \leq N \) and \( M_1(i-1)+1 \leq j \leq M_1i \). Equality holds for all \( i \) and \( j \) if and only if \( 1/F(s) \) is lossless positive-real.

From the above discussion, we finally present the following theorem that summarizes the statements of Propositions 3 and 4.

**Theorem 2:** The values of the second-order modes of linear continuous-time systems are decreased under general positive-real transformations. That is, given a stable transfer function \( H(s) \) and a positive-real function \( 1/F(s) \), the values of the second-order modes of \( H(F(s)) \) are smaller than those of \( H(s) \). In the special case when \( 1/F(s) \) is lossless, the second-order modes of \( H(F(s)) \) are the same as those of \( H(s) \).

As stated above, Theorem 2 is the generalized version of Theorem 1. Theorems 1 and 2 enable us to know the relationship between the second-order modes of \( H(s) \) and those of \( H(F(s)) \) for all positive-real functions \( 1/F(s) \).

5. Numerical Examples

This section gives numerical examples to demonstrate Theorems 1 and 2. Consider the following single-input/single-output transfer function \( H(s) \) of order 2:

\[
H(s) = \frac{1.4314}{s^2 + 1.4256s + 1.5162} \tag{64}
\]

This transfer function has a lowpass magnitude response shown in Fig. 1. The second-order modes of this system is calculated as

\[
(\theta_1, \theta_2) = (0.7071, 0.2351). \tag{65}
\]

For this system, we apply a lossless positive-real transformation \( s^{-1} \leftarrow 1/F_1(s) \) and a positive-real transformation \( s^{-1} \leftarrow 1/F_2(s) \), where \( 1/F_1(s) \) and \( 1/F_2(s) \) are respectively given by

\[
\frac{1}{F_1(s)} = \frac{s^3 + 5s}{2s^2 + 6} \tag{66}
\]

\[
\frac{1}{F_2(s)} = \frac{1}{F_1(s)} + 0.1 = \frac{s^3 + 0.2s^2 + 5s + 0.6}{2s^2 + 6}. \tag{67}
\]

The above transformations yield the following transformed systems, respectively:

\[
H(F_1(s)) = \frac{N_1(s)}{D_1(s)} \tag{68}
\]

\[
H(F_2(s)) = \frac{N_2(s)}{D_2(s)} \tag{69}
\]

where

\[
N_1(s) = 0.9441s^6 + 9.4407s^4 + 23.6018s^2 \tag{70}
\]

\[
D_1(s) = s^6 + 1.8805s^5 + 12.6382s^4 + 15.0439s^3
+ 40.8290s^2 + 28.2074s + 23.7436 \tag{71}
\]

\[
N_2(s) = 0.9441s^6 + 0.3776s^5 + 9.4785s^4 + 3.0210s^3
+ 23.8283s^2 + 5.6644s + 0.3399 \tag{72}
\]

\[
D_2(s) = s^6 + 2.2805s^5 + 13.0543s^4 + 18.2439s^3
+ 43.3256s^2 + 34.2074s + 27.4885. \tag{73}
\]

The magnitude responses of \( H(s), H(F_1(s)) \) and \( H(F_2(s)) \) are given in Fig. 1.

Now, from (68), the second-order modes of \( H(F_1(s)) \) are calculated as

\[
(\bar{\theta}_{1,1}, \bar{\theta}_{1,2}, \bar{\theta}_{1,3}, \bar{\theta}_{1,4}, \bar{\theta}_{1,5}, \bar{\theta}_{1,6})
= (0.7071, 0.7071, 0.7071, 0.2351, 0.2351, 0.2351). \tag{74}
\]

Comparing (65) with (74), we see that \( \bar{\theta}_{1,1} = \bar{\theta}_{1,2} = \bar{\theta}_{1,3} = \theta_1 \) and \( \bar{\theta}_{1,4} = \bar{\theta}_{1,5} = \bar{\theta}_{1,6} = \theta_2 \), which shows that the second-order modes of \( H(s) \) are invariant under the lossless positive-real transformation \( s^{-1} \leftarrow 1/F_1(s) \).

Similarly, the second-order modes of \( H(F_2(s)) \) are calculated as

\[
(\bar{\theta}_{2,1}, \bar{\theta}_{2,2}, \bar{\theta}_{2,3}, \bar{\theta}_{2,4}, \bar{\theta}_{2,5}, \bar{\theta}_{2,6})
= (0.7071, 0.7071, 0.7071, 0.2351, 0.2351, 0.2351). \tag{75}
\]
\[
(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6)
= (0.6227, 0.6227, 0.6227, 0.1568, 0.1568, 0.1568)
\]
which shows \(\theta_1, \theta_2, \theta_3 < \theta_4\) and \(\theta_4, \theta_5, \theta_6 < \theta_2\). From this result we see that the second-order modes of \(H(s)\) are decreased under the general positive-real transformation \(s^{-1} \leftarrow 1/F_2(s)\).

6. Concluding Remarks

This paper has discussed the behavior of the second-order modes of linear continuous-time systems under positive-real transformations. We have proved that the second-order modes are invariant under any lossless positive-real transformation. Furthermore, we have discussed the case of general positive-real transformations, and proved that the general transformations decrease the values of the second-order modes. Since the second-order modes play significant roles in many practical and theoretical issues of linear system theory, our results will bring new insights into the study of linear dynamical systems and their applications.

Finally, we address the importance and significance of the discussion presented in this paper. As mentioned earlier, our results are parallel to the theory of discrete-time lossless transformations [12] and discrete-time general transformations [16]. In some cases, parallel results in the continuous-time case can be easily obtained from the results in the discrete-time case by means of simple tools such as bilinear transformation. However, the results that has been presented in this paper cannot be directly obtained from the discrete-time counterpart. The reason for this lies in the fact that the lossless and general transformations of continuous-time systems are not related to those of discrete-time systems by the bilinear transformation — in the continuous-time case, lossless and general transformations are defined as positive-real functions, whereas the discrete-time lossless and general transformations make use of bounded-real functions. In addition, transformations in the continuous-time case include improper functions as well as proper ones, whereas in the discrete-time case improper transformations do not exist. Hence, it is impossible to derive the results of this paper by simply extending the results of [12] and [16] to the continuous-time case, and thus a separate discussion from the discrete-time version is required for derivation of the continuous-time results. Furthermore, it is clear that discussions for continuous-time systems are more essential to the field of linear system theory than those for discrete-time systems. These facts show the importance and significance of the analysis presented in this paper.

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