Abstract—The limits of application, tuning accuracy and sensitivity of variable IIR digital filters realized as a cascade of several identical sub-filters are investigated in this paper. Sub-filters of first and second-order are realized with bilinear and biquadratic sections with independent tuning, developed by the authors, while sub-filters of higher order are realized as parallel allpass structures using first and second-order allpass sections with minimized sensitivities. It is shown that filters so designed are having higher accuracy and wider ranges of tuning compared to other known variable filters and are behaving much better in a limited word-length environment.

I. INTRODUCTION

Variable IIR filters are usually designed by applying the spectral (allpass) transformations of Constantinides (TC) [1], [2] on some normalized lowpass (LP) prototypes and injecting thus one or two additional parameters used for tuning of the cutoff frequency of the LP/HP (highpass) filters or the central frequency and the bandwidth (BW) of the bandpass/bandstop (BP/BS) filters. But when the prototypes are IIR filters, delay-free loops appear after the TC. Due to the attempts to eliminate these delay-free loops, no precise, without limitations, real-time tuning of IIR filters is known until now – all methods are approximate and valid only in a narrow range of values of the tuned parameter and over some limited frequency range. The best known method (called MNR-method after the names of the authors Mitra, Neuvo and Roivainen) [3] is based on truncated Taylor series expansions, applied on real or complex coefficient parallel-allpass-structure, but we have shown in [4] that its magnitude characteristics are degrading even when the filter is tuned over a very limited frequency range. We have increased considerably the range and the accuracy of tuning by introducing a sensitivity minimization as an additional design step [4]. We have developed also a new approach [5], based on a cascaded connection of several identical sub-filters. It permits an easy tuning of the cutoff frequency of the LP filter without having to use TC and truncated Taylor series expansions when using sub-filters of first or second order. We have developed and investigated [5], [6], [7] such tunable sub-filter structures (of first and second order) with independent or simultaneous tuning of their frequency parameters. In this work we propose to realize the sub-filters of higher than second order as parallel allpass structures. Then we investigate the applicability, the sensitivity and the tuning accuracy of the filters designed according to our approach and compare them with MNR-filters.

II. APPROXIMATION USING A PRODUCT OF SEVERAL EQUAL TRANSFER FUNCTIONS

A. Synopsis of the method

Our new method of design and realization of variable digital filters (VDF) is based on the usage of several cascaded identical filter blocks, each of them providing a very simple tuning of a given frequency parameter by varying a single multiplier coefficient. We are concerned with development of approximation procedures meeting only lowpass filter specifications. Variable BP/BS filters are obtained then by applying the constrained TC [1][2] on the variable LP filter. This transformation provides also an independent tuning of the central frequency, while the tuning of the cutoff frequency of the prototype LP filter is varying the BW of these BP/BS filters. The magnitude specifications of the desired LP variable filter are: pass-band (PB) from 0 to \( \omega_p \) (for digital filters) or \( \Omega_p \) (for analog), stop-band (SB) from \( \omega_s \) or \( \Omega_s \) to infinity, maximum variation of the PB attenuation \( A_p \), dB and minimum SB attenuation \( A_s \), dB.
we have to find a total transfer function (TF) \( H(z) \) (digital) or \( T(s) \) (analog) represented as a product of \( N \) equal individual TFs \( H_i(z) \) or \( T_i(s) \), each of them of order \( n \):

\[
H(z) = H_1^N(z), \quad T(s) = T_1^N(s),
\]

\[
H_i(z) = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_n z^{-n}}{1 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_n z^{-n}},
\]

\[
T_i(s) = \frac{a_0 + a_1 s + a_2 s^2 + \ldots + a_m s^m}{b_0 + b_1 s + b_2 s^2 + \ldots + b_n s^n}, \quad m \leq n.
\]

These TFs might be of Butterworth, Chebyshev or elliptic type and in the process of design we have to determine the minimal number \( N \) of the individual TFs \( H_i(z) \) or \( T_i(s) \), necessary to meet the specifications with given (selected) type (maximally flat, equiripple or other) and order \( n \). A step-by-step design procedure for this is given in [5], it is performed in the analog domain and is using the popular in the classical filter theory Characteristic function \( k(\Omega) \).

**B. Limitations of the method**

An approximation using \( N \) equal terms is far from optimal and there are many limitations that have to be clearly defined. These limitations depend on the type of the individual TF and on the selected approximation parameters. It might be even impossible to meet some difficult filter specifications no matter how high the number \( N \) is taken. These limitations are investigated in details in [5]. Here we need a more general evaluations taking into account the order \( n \) of the individual TFs and some simple parameter describing how difficult the specifications are. We choose to use the "rectangularity coefficient" \( r \) (calculated in \( s \)- or in \( z \)-domain) as such a parameter:

\[
r = \frac{\Omega_s}{\Omega_p} \quad \text{or} \quad r = \frac{\omega_p \tau}{2} \left/ \frac{\omega_s \tau}{2} \right., \quad (4)
\]

where \( \tau \) is the sampling interval and \( r \) is taking values within the limits \( 1 \leq r < \infty \) with \( r=1 \) for an ideal LP filter. If \( A_r \) is the highest value that \( A_r \) can achieve at its points of minimum (in the stop-band) for a given approximation and values of \( n \) while \( N \to \infty \), we can derive very handy formulae connecting only \( A_p, r \) and \( A_r \) max and permitting the most general possible way of comparing different approximations. For Butterworth-type of individual TF we get (starting from the results in [5])

\[
A_r \text{max} = A_p r^{2n}, \quad (5)
\]

which is used to calculate the curves shown in Fig. 1a. It is clear that the ratio \( A_r \text{max} / A_p \) cannot exceed the value of 100 even for very easy filter specifications (\( r=3 \), for example) if we use an unlimited number of Butterworth-type second-order sections as individual sub-filters.

If the individual TFs are of Chebyshev type, we obtain the following very general expression

\[
A_r \text{max} = A_p \chi^2(n \text{Arch} \, r), \quad (6)
\]

which is producing a family of curves shown in Fig. 1b. It is seen that in order to obtain a filter with quite a high selectivity (\( r<2 \)), it will be necessary to use individual sub-filters of order higher than second even when these sub-filters are with Chebyshev type of TF. These filters are, on the other hand, much more capable, compared to the filters with Butterworth individual TF (which is very easy to anticipate).

It is impossible to derive a single compact formula for the case of elliptic type of individual TFs. In [5] we have derived very general expressions (different for even and odd \( n \))

\[
A_r \text{max} = A_p r^4 \left/ \frac{K(1/r)}{2} \right. \quad \text{for } n = 2, \quad (7)
\]

\[
A_r \text{max} = A_p r^8 \left/ \frac{K(1/r)}{3} \right. \quad \text{for } n = 3, \quad (8)
\]

\[
A_r \text{max} = A_p r^{12} \left/ \frac{K(1/r)}{5} \right. \quad \text{for } n = 5, \quad (9)
\]

where \( sn \) is the Jacobi's "sinus elliptic" function and \( K(1/r) \) the function of Jacobi, calculated as a complete elliptic integral of first kind (see [5] for details). It is seen from Fig. 2a that almost any kind of difficult filter specifications will be met if elliptic transfer functions with order higher than second are used to realize the individual sub-filters.

In Fig. 2b the upper limits of the stop-band attenuation, achievable with identical first- (IFOS) and second-order sections (ISOS), are given in order to have a base for comparisons. These limits are, in fact, much lower because instead of \( N \to \infty \), we are using only several IFOS or ISOS.
III. REALIZATION OF HIGH TUNING ACCURACY VARIABLE SUB-FILTERS

High tuning accuracy means a possibility to tune the frequency parameters (cutoff frequencies and the BW) to any given value of these parameters. But in the limited word-length environment the possible TF poles and zeros positions in some areas of the z-plane are very sparse (which is easily illustrated with the density of the poles distribution) and so are the possible values of the frequency parameters over some frequency ranges. Low pole-density, on the other hand, means high pole-sensitivity, so the problem of the tuning accuracy improvement is directly connected to the minimization of the sensitivity.

A. First and second-order sections

We have shown in [5] that the best possible first-order LP/HP variable filter section, suitable for narrow-band IFOS realizations with poles near z=1, is the one shown in Fig. 3a. We have developed several biquadratic sections [5], [6], [7] for ISOS VDFs and the best of them (called BQ3), is shown in Fig. 3b. Its elliptic and LP TFs are

\[ H_E(z) = \frac{a_0}{1 - z^{-1} + c z^{-2}} \]

and

\[ H_{LP}(z) = c(1 - d)(1 + z^{-1})^{-1} \frac{D(z)}{D(z)} \]

BQ3 is providing, as seen from (10), an independent tuning of its parameters without having to use any Taylor series expansions. It was shown in [6], [7], that it is providing also a simultaneous tuning of the pole- and the zero-frequencies of the elliptic TF by trimming only the multiplier coefficient c.

B. Higher order subfilters

For higher than second-order sub-filters we have to accept the MNR approach (Fig. 4) [3] but only for realization of limited order n. In Fig. 4b it is shown how a coefficient \( a_1 \) is turned to variable \( (a_1)_{\text{var}} = a_1 + \alpha K \) by adding a parallel branch, containing a variable coefficient \( \alpha \) and an additional coefficient \( K \) properly calculated by using Taylor series expansions [3]. It provides an easy tuning of the cutoff frequency of the LP individual sub-filters by varying a single multiplier coefficient. But as our structure is a cascade of several low-order sub-filters (even though obtained as parallel-allpass-structures), it has much lower SB sensitivity, compared to that of the totally parallel allpass structure, which is behaving really badly, as shown in [4]. And instead of using the most popular in the literature first- (called MH) (Fig. 5a) and second-order (called MH2B) (Fig. 5c) allpass sections, we have developed some very-low-sensitivity (for poles near z=1) first- (called ST) (Fig. 5b) and second-order (called LS) (Fig. 5d) sections, realizing the following TFs:

\[ H(z)_{ST} = \frac{1 - z^{-1} - \alpha a_0}{1 - z^{-1} (1 - \alpha a_0)} \]

and

\[ H(z)_{LS} = \frac{1 - c_2 z^{-1} + \alpha a_0}{1 - (2 - 2c_1 - c_2) z^{-1} + (1 - c_2) z^{-2}} \]

IV. SENSITIVITY INVESTIGATIONS

If \( \Theta_A \) and \( \Theta_B \) are the phase responses of the individual first- and second-order allpass sections in the two branches of Fig. 4a, we can derive the following expressions for the worst-case sensitivities for some specific points of the magnitude, like the cutoff frequency \( \omega_0 \) and the one of minimal attenuation \( \omega_{\text{min}} \) (for elliptic approximation) in the SB:

\[ wS_b[H(\omega_0)] = \frac{1}{2} \sum_{i=1}^{L} \sum_{j=1}^{M} \phi_{\xi_i}(\omega_0) \Theta_{B_j}(\omega_0) + \sum_{i=1}^{L} \phi_{\xi_i}(\omega_{\text{min}}) \Theta_{B_j}(\omega_{\text{min}}) \]

(14)

\[ wS_b[H(\omega_{\text{min}})] = \frac{1}{2} \sum_{i=1}^{L} \sum_{j=1}^{M} \phi_{\xi_i}(\omega_{\text{min}}) \Theta_{B_j}(\omega_{\text{min}}) + \sum_{i=1}^{L} \sum_{j=1}^{M} \phi_{\xi_i}(\omega_{\text{min}}) \Theta_{B_j}(\omega_{\text{min}}) \]

(15)

where \( M \) depends on \( A_s \) in the specifications \( A_s = 30\text{dB}, 50 \text{ for } 40 \text{dB} \) and \( 158 \text{ for } 50 \text{dB} \). The proper selection of these sections obviously is critically important.

We have investigated, using the program PANDA [8], the worst-case sensitivities of two MNR-realizations of same specifications \( F_{\omega_0} = 0.01, F_{\omega_{\text{min}}} = 0.03, A_{s} = 1\text{dB}, A_{s} = 35 \text{dB} \) – one with MH, MH2B and one with our sections (ST and LS – Figs 5b, d). It is seen from the result, shown in Fig. 6a, that the usage of proper sections is reducing the SB sensitivity
more than 60 times. Then we have realized a VDF with the same specifications, but using our new approach (cascade ISOS in this case), employing the BQ3 section (Fig. 3b) and investigated the sensitivity. The results shown in Fig. 6b demonstrate a startling reduction of more than 300 times, which makes our approach the best possible. It is clear that really low SB sensitivities, i.e., very high tuning accuracy, are possible only if IFOS or ISOS realizations are employed. Only when high filter selectivity (impossible to meet with IFOS or ISOS) is required, we have to use MNR-based third- or fifth-order sub-filters (it is impossible to realize even-order LP sub-filters with real coefficients and sub-filters of order higher that 5 are impractical, because we lose then all the merits of the cascade realization), but with our very low sensitivity allpass sections of Fig. 5b, d. We have employed an indirect approach for a comparative study of the sensitivities of such filters. For quite specific situations \( F_p=0.01, F_{sw}=0.03, A_p=2\text{dB}, A_{sw}=55 \text{ dB} \) and Butterworth approximation we have obtained 7th order TF and realized it as MNR VDF using MH and MH2B (Figs. 5a, c) sections. Then a cascaded realization with third-order sub-filters was designed (using the sections of Figs. 5b, d) for the same specifications, which produced \( N=3 \) or total order 9. Both filters have been tuned with factor \( \alpha=0.033 \) (Fig. 4b) and then simulated with coefficients quantized to different word-length \( B \) (supposing “canonic sign-digit code”). The results are shown in Fig. 7. The MNR-filter characteristics (Fig. 7a) are destroyed even with \( B=7\text{bit} \) – the attenuation is changed from Butterworth type to something like elliptic and is getting some SB minimum of about 15 dB which is far below the limit of 55 dB. Our filter is behaving perfectly even with \( B=3 \) (Fig. 7b) and is changing slightly for \( B=2\text{bit} \).

**V. EXPERIMENTS**

The range of tuning of the filters described in Fig. 7 has been studied through simulations and the results are shown in Fig. 8. The MNR filter is changing its type (from Butterworth to non-polynomial) for \( \alpha > 0.11 \) and for \( \alpha > 0.11 \) the SB specifications are already violated, while our filter is smoothly tuned from \( \alpha=0.3 \) to \( \alpha=0.3 \), covering thus very wide frequency range without any magnitude degradation.

**VI. CONCLUSIONS**

A new type of variable IIR filters realized as a cascade of \( N \) identical sub-filters have been investigated in this paper. The general limitations of the method and the sensitivities of the proposed structures are studied and it is shown that filters so designed are having higher accuracy and wider range of tuning compared to other known variable filters and they are behaving much better in a limited word-length environment at a price of a slight increase of their order. All theoretical results and the superiority of the new method are verified experimentally.

**REFERENCES**


