A Novel Property of the Second-Order Modes of Discrete-Time Systems under Variable Transformation

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Abstract—This paper derives a novel property of the second-order modes of discrete-time systems under variable transformation. This is the generalization of the theory presented by Mullis and Roberts, who discovered the invariance of the second-order modes under allpass transformation (variable transformation of lossless bounded-real function) and the relationship of this invariance to the roundoff noise of digital filters. This paper proves that the values of the second-order modes are decreased under any variable transformation of bounded-real function. The proof is achieved with the help of the bounded-real lemma and formulation of the controllability and observability Gramians of transformed discrete-time systems.

I. INTRODUCTION

The second-order modes are defined as the positive square roots of the eigenvalues of the matrix product of the controllability and observability Gramians of linear state-space systems. They play crucial roles in signal processing theory and control theory as follows:

2) The second-order modes determine the upper bound of the approximation error due to balanced model reduction [3], [4].

In addition to the above-mentioned theoretical and practical importance, the second-order modes have interesting and important properties. In particular, Mullis and Roberts’ theory on the invariance of the second-order modes under frequency transformation [5] has attracted many researchers because this property shows the excellent noise performance of narrowband digital filters with optimal realization and can be applied to the realization of variable digital filters with high performance [6]. This theory has also been discussed in the case of continuous-time systems and two-dimensional digital filters [7], [8].

However, in the Mullis and Roberts’ theory [5], the function for frequency transformation was restricted to lossless bounded-real function (allpass function). Therefore, there has been no research on the analysis of the second-order modes under general variable transformation of bounded-real function.

In this paper, we generalize the Mullis and Roberts’ theory. That is, we investigate the property of the second-order modes of discrete-time systems under general variable transformation and prove that the values of the second-order modes are decreased under any variable transformation of bounded-real function.

The organization of this paper is as follows. Section II gives a brief introduction of the discrete-time state-space systems, Gramians and second-order modes. Section III addresses the theory on variable transformation and the bounded-real lemma, which are frequently used in our main discussion. Section IV presents our main result; the property of the second-order modes under general variable transformation is revealed. Section V gives a numerical example in order to verify our theory. Section VI gives conclusion.

II. THE SECOND-ORDER MODES OF DISCRETE-TIME SYSTEMS

Consider an $N$-th order stable single-input/single-output discrete-time system $H(z)$ described by the following state-space equations:

\[ x(n + 1) = Ax(n) + bu(n) \]  \hfill (1)

\[ y(n) = cx(n) + du(n) \]  \hfill (2)

where $u(n)$, $y(n)$ and $x(n)$ are the scalar input, the scalar output and the state vector with the size of $N \times 1$, and the matrices $A$, $b$, $c$ and $d$ are coefficient matrices with appropriate size. This system is assumed to be controllable and observable. The coefficient matrices $(A, b, c, d)$ and the transfer function $H(z)$ are related as

\[ H(z) = d + c(zI_N - A)^{-1}b \]  \hfill (3)

where $I_N$ is the $N \times N$ identity matrix.

For the system $(A, b, c, d)$, the solutions $K$ and $W$ of the following Lyapunov equations are called the controllability Gramian and the observability Gramian, respectively:

\[ K = AA^T + bb^T \]  \hfill (4)

\[ W = A^TWA + c^Tc. \]  \hfill (5)

These Gramians $K$ and $W$ are symmetric and positive definite because the system $(A, b, c, d)$ of $H(z)$ is assumed to be stable, controllable and observable. Then, the eigenvalues $\theta_1^2, \theta_2^2, \ldots, \theta_N^2$ of the matrix product $KW$ are all positive real.
The positive square roots \( \theta_1, \theta_2, \cdots, \theta_N \) of the eigenvalues are called the second-order modes of the system [3], [5].

It should be noted that the Gramians depend on realization of the system, while the second-order modes depend only on the transfer function. In the literatures on control system theory, the second-order modes are also called Hankel singular values because the eigenvalues of \( KW \) are equal to the singular values of the Hankel operator of \( H(z) \).

III. VARIABLE TRANSFORMATION AND BOUNDED-REAL LEMMA

In this section, the theory on variable transformation and the bounded-real lemma are introduced. They play central roles in the derivation of our main result.

A. Formulation of variable transformation in terms of the state-space equations

Consider the following discrete-time system

\[ G(z) = H(F(z)) = H(z)|_{z=1} = 1/F(z) \]

where \( H(z) \) and \( 1/F(z) \) are an \( N \)-th order and an \( M \)-th order stable discrete-time system, respectively, and thus the composite system \( G(z) \) has the order of \( MN \). To guarantee the stability of \( G(z) \), \( 1/F(z) \) is assumed to be bounded-real, that is, \( |1/F(e^{\omega t})| \leq 1 \) for all \( \omega \).

In Ref. [5], the following important lemma is established for the formulation of \( H(F(z)) \) in terms of the state-space equations.

**Lemma 1:** Let \( (A, b, c, d) \) and \( (\alpha, \beta, \gamma, \delta) \) be a state-space representation of \( H(z) \) and \( 1/F(z) \), respectively. Then, a state-space representation of the composite system \( G(z) = H(F(z)) \) is described by

\[ G(z) = D + C(zI_{MN} - \mathcal{A})^{-1}B \]

where \( I_{MN} \) is the \( MN \times MN \) identity matrix and the coefficients \( A, B, C \) and \( D \) are given as

\[ A = I_N \otimes \alpha + [A(I_N - \delta A) - \mathcal{A}] \otimes (\beta \gamma) \]
\[ B = [(I_N - \delta A)^{-1}b] \otimes \beta \]
\[ C = [c(I_N - \delta A)^{-1}] \otimes \gamma \]
\[ D = d + \delta c(I_N - \delta A)^{-1}b. \]

In the above equations, \( \otimes \) denotes the Kronecker product for matrices.

B. Bounded-real lemma [9], [10]

**Lemma 2:** Let \( (\alpha, \beta, \gamma, \delta) \) parameterize a state-space representation of a discrete-time system \( 1/F(z) \), that is,

\[ 1/F(z) = \delta + \gamma(zI_M - \alpha)^{-1}\beta \]

where \( I_M \) is the \( M \times M \) identity matrix. If \( 1/F(z) \) is bounded-real, there exist a real vector \( l \), a real scalar \( w_1 \), and a real symmetric positive definite matrix \( P \) such that

\[ \alpha^lP\alpha + \gamma^l\gamma + \ell^lw_1 = P \]
\[ \beta^lP\beta + \delta^2 + w_1^2 = 1 \]
\[ \alpha^lP\beta + \gamma^l\delta + \ell^lw_1 = 0. \]

Note that Eq. (13) is equivalent to the following discrete-time bounded-real Riccati equation

\[ P - \alpha^lP\alpha - \gamma^l\gamma - (\alpha^lP\beta + \gamma^l\delta) \\
\times (1 - \delta^2 - \beta^lP\beta)^{-1}(\alpha^lP\beta + \gamma^l\delta)^l = 0. \] (16)

IV. THE PROPERTY OF THE SECOND-ORDER MODES UNDER VARIABLE TRANSFORMATION

In this section, we present our main result; we reveal the property of the second-order modes under variable transformation. Before the main discussion, we give the following lemma.

**Lemma 3:** Let \( (\alpha, \beta, \gamma, \delta) \) be a state-space representation of a bounded-real system \( 1/F(z) \). Then, there exist a real vector \( m \), a real scalar \( w_2 \), and a real symmetric positive definite matrix \( Q \) such that

\[ \alpha Q \alpha^l + \beta \beta^l + mm^l = P \]
\[ \gamma Q \gamma^l + \delta^2 + w_2^2 = 1 \]
\[ \alpha Q \gamma^l + \beta \delta + mw_2 = 0. \]

Moreover, the matrix \( Q \) satisfies \( Q = P^{-1} \), where \( P \) is the positive definite matrix given in Lemma 2.

**Proof:** Eqs. (17)–(19) are obvious because they are completely dual to Eqs. (13)–(15). \( Q = P^{-1} \) can be proved by applying the bilinear transformation to Lemma 1 of Ref. [11]. Details are omitted here. \( \square \)

We are now ready to discuss the property of the second-order modes under variable transformation. First, we give the following lemma regarding the Gramians of the composite system \( H(F(z)) \).

**Lemma 4:** Let \( K \) and \( W \) be the controllability and observability Gramian of \( H(z) = d + c(zI_N - A)^{-1}b \), and let \( K \) and \( \mathcal{W} \) be the controllability and observability Gramian of \( H(F(z)) = D + C(zI_{MN} - \mathcal{A})^{-1}B \). That is, \( K \) and \( W \) satisfy Eqs. (4) and (5) and \( K \) and \( W \) satisfy the following Lyapunov equations:

\[ K = AKKA^l + BB^l \]
\[ W = A^lWAA + C^lC. \]

Then, for any bounded-real system \( 1/F(z) \), the following inequalities hold:

\[ \mathcal{W} \leq W \otimes P \]
\[ K \leq K \otimes Q \]

where \( P \) and \( Q \) are the positive definite matrices given in Lemma 2 and Lemma 3, respectively.

Before giving the proof of the above lemma, we introduce the properties for the Kronecker product \( \otimes \) as follows [12]:

\[ (A \otimes B)(C \otimes D) = (AC) \otimes (BD) \]
\[ (A + B) \otimes (C + D) = A \otimes C + A \otimes D + B \otimes C + B \otimes D \]
\[ (A \otimes B)^{-1} = A^{-1} \otimes B^{-1} \]
\[ (A \otimes B)^l = A^l \otimes B^l. \]
In the following proof, we make frequent use of these properties.

Proof: In this paper we restrict ourselves to the proof of (22) because the proof of (23) is completely dual to the present one.

First, consider the following matrix

\[
X = [(I_N - \delta A) \otimes I_M]^t \cdot [(W \otimes P) - \mathcal{A}^t(W \otimes P)A - C^tC] \cdot [(I_N - \delta A) \otimes I_M].
\]  
(28)

Substituting Eqs. (8) and (10) into Eq. (28) yields

\[
X = [(I_N - \delta A^t)W(I_N - \delta A)] \otimes P \\
- [(I_N - \delta A^t)WtI]$A - (\delta^tP)(\alpha^tP\alpha) \\
- [A^tWtI - 1 - w_1^t1) \gamma ] \\
- (c^t c) \otimes (\gamma \gamma ).
\]  
(29)

With the help of Eqs. (13)–(15), Eq. (29) is simplified as follows:

\[
X = [(I_N - \delta A^t)W(I_N - \delta A)] \otimes (\gamma \gamma + t^t1) \\
- (WtI - \delta A^tWtI) \otimes (-\delta^t \gamma - w_1^t1) \\
- (A^tWtI - \delta A^tWtI) \otimes (-\delta^t \gamma - w_1^t1) \\
- (A^tWtI - 1 - \delta^t \gamma ) \\
- (c^t c) \otimes (\gamma \gamma ) \\
= (WtI - A^tWtI - c^t c) \otimes (\gamma \gamma ) \\
+ (w_1^tA^tWtI) \otimes (\gamma \gamma ) \\
+ [(I_N - \delta A) \otimes I_M]^{-1}Y^tY[(I_N - \delta A) \otimes I_M]^{-1}. 
\]  
(30)

where we let

\[
Y = W^t(I_N - \delta A) \otimes I + (w_1^tW^tA) \otimes \gamma.
\]  
(31)

Therefore, from Eqs. (28) and (30) it follows that

\[
(W \otimes P) - \mathcal{A}^t(W \otimes P)A - C^tC \\
= [(I_N - \delta A) \otimes I_M]^{-1}Y^tY[(I_N - \delta A) \otimes I_M]^{-1}. 
\]  
(32)

Subtracting Eq. (21) from Eq. (32) gives

\[
(W \otimes P - \mathcal{W}) - \mathcal{A}^t(W \otimes P - \mathcal{W})A \\
= [(I_N - \delta A) \otimes I_M]^{-1}Y^tY[(I_N - \delta A) \otimes I_M]^{-1} \\
geq 0.
\]  
(33)

Since the eigenvalues of \( \mathcal{A} \) lie inside the unit circle because of the stability of \( H(F(z)) \), (33) has a unique solution \( W \otimes P - \mathcal{W} \geq 0 \). This proves \( \mathcal{W} \leq W \otimes P \). The proof of \( \mathcal{K} \leq \mathcal{K} \otimes \mathcal{Q} \) can be achieved in a similar way by using Eqs. (8), (9) and (17)–(19).

Lemma 4 gives the theorem on the second-order modes under variable transformation as follows.

**Theorem 1:** Let \( \theta_1 \geq \theta_2 \geq \cdots \geq \theta_N \) be the second-order modes of an \( N \)-th order discrete-time system \( H(z) \), and let \( \bar{\theta}_1 \geq \bar{\theta}_2 \geq \cdots \geq \bar{\theta}_{MN} \) be the second-order modes of an \( MN \)-th order discrete-time system \( H(F(z)) \), where \( 1/F(z) \) is a bounded-real \( M \)-th order system. Then,

\[
\bar{\theta}_i \leq \theta_j \tag{34}
\]

where

\[
i = M(j - 1) + 1, M(j - 1) + 2, \cdots, Mj
\]

\[
 j = 1, 2, \cdots, N.
\]

In (34), equality holds if and only if \( |1/F(e^{j\omega})| = 1 \) for all \( \omega \), that is, \( 1/F(z) \) is a lossless bounded-real function (allpass function).

Proof: From Lemma 4, it is shown that

\[
\mathcal{K} \mathcal{W} \leq (\mathcal{K} \mathcal{W}) \otimes (\mathcal{Q} \mathcal{P}).
\]  
(37)

Since there exists \( Q = P^{-1} \) from Lemma 3, (37) becomes

\[
\mathcal{K} \mathcal{W} \leq (\mathcal{K} \mathcal{W}) \otimes I_M.
\]  
(38)

Let \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{MN} \) be the eigenvalues of \( \mathcal{K} \mathcal{W} \), and let \( \lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_{MN} \) be the eigenvalues of \( (\mathcal{K} \mathcal{W}) \otimes I_M \). Then, from (38) it is easily seen that

\[
\lambda_i \leq \lambda'_i \tag{39}
\]

for \( i = 1, 2, \cdots, MN \). Moreover, the eigenvalues \( \lambda'_1, \lambda'_2, \cdots, \lambda'_{MN} \) are actually \( M \) copies of the eigenvalues of \( \mathcal{K} \mathcal{W} \). Thus, letting \( \lambda''_1 \geq \lambda''_2 \geq \cdots \geq \lambda''_N \) be the eigenvalues of \( \mathcal{K} \mathcal{W} \), (39) can be rewritten as

\[
\lambda_i \leq \lambda''_i \tag{40}
\]

where

\[
i = M(j - 1) + 1, M(j - 1) + 2, \cdots, Mj
\]

\[
 j = 1, 2, \cdots, N.
\]

This shows (34)–(36). The proof in the case of \( |1/F(e^{j\omega})| = 1 \) for all \( \omega \) can be seen in Ref. [5].

Theorem 1 shows that the values of the second-order modes are decreased under any variable transformation of bounded-real function. This is the generalized version of the theory of Mullis and Roberts [5]. The simplest case in Theorem 1 is \( M = 1 \), where \( \bar{\theta}_i \leq \theta_i \) for \( i = 1, 2, \cdots, N \).

V. A Numerical Example

This section gives a numerical example to verify our theory. Consider the following transfer function of a 2nd-order discrete time system \( H(z) \):

\[
H(z) = \frac{0.0931 + 0.1862z^{-1} + 0.0931z^{-2}}{1 - 1.0349z^{-1} + 0.4293z^{-2}}.
\]  
(43)

A state-space representation \((A, b, c, d)\) of this system is given as

\[
\begin{pmatrix}
A & b \\
c & d
\end{pmatrix} = \begin{pmatrix}
1.0349 & -0.4293 \\
1 & 0 \\
0.2825 & 0.0531
\end{pmatrix}. 
\]  
(44)
This system has the following controllability Gramian, observability Gramian, and second-order modes:

\[
K = \begin{pmatrix}
2.5767 & 1.8656 \\
1.8656 & 2.5767
\end{pmatrix},
\]

\[
W = \begin{pmatrix}
0.2689 & -0.0731 \\
-0.0731 & 0.0524
\end{pmatrix},
\]

\[(\theta_1, \theta_2) = (0.7071, 0.2351).\]  \hspace{1cm} (47)

For this system, we apply the variable transformation (6) with

\[
\frac{1}{F(z)} = \frac{0.1181 + 0.2363z^{-1} + 0.1181z^{-2}}{1 - 0.7089z^{-1} + 0.1815z^{-2}}.
\]  \hspace{1cm} (48)

A state-space representation \((\alpha, \beta, \gamma, \delta)\) of \(1/F(z)\) is given as

\[
\begin{pmatrix}
\alpha \\
\beta \\
\gamma \\
\delta
\end{pmatrix} = \begin{pmatrix}
0.7089 & -0.1815 & 1 \\
0.3200 & 0.0967 & 0.1181
\end{pmatrix}.
\]  \hspace{1cm} (49)

The resultant composite system \(H(F(z))\) and its state-space representation \((\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})\) are obtained as follows:

\[
H(F(z)) = \frac{0.1317-0.1113z^{-1}+0.0941z^{-2}-0.0298z^{-3}+0.0095z^{-4}}{1-0.7089z^{-1}+0.1815z^{-2}}.
\]  \hspace{1cm} (50)

\[
\mathcal{A} = \begin{pmatrix}
0.1063 & -0.0738 & -0.1555 & -0.0470 \\
1 & 0 & 0 & 0 \\
0.3622 & 0.1094 & 0.6905 & -0.1870 \\
0 & 0 & 1 & 0
\end{pmatrix},
\]  \hspace{1cm} (51)

\[
\mathcal{B} = \begin{pmatrix}
1.1316 \\
0 \\
0.1337 \\
0
\end{pmatrix},
\]  \hspace{1cm} (52)

\[
\mathcal{C} = \begin{pmatrix}
0.1046 & 0.0316 & 0.0117 & 0.0035
\end{pmatrix},
\]  \hspace{1cm} (53)

\[
\mathcal{D} = 0.3137.
\]  \hspace{1cm} (54)

The magnitude responses of \(H(z), 1/F(z)\) and \(H(F(z))\) are given in Figure 1. From Eqs. (51)–(54), the controllability Gramian, observability Gramian, and second-order modes of this system are calculated as follows:

\[
\mathcal{K} = \begin{pmatrix}
5.5586 & 4.8045 & 3.9991 & 2.9988 \\
4.8045 & 5.5586 & 4.7395 & 3.9991 \\
3.9991 & 4.7395 & 4.5622 & 4.3110 \\
2.9988 & 3.9991 & 4.3110 & 4.5622
\end{pmatrix},
\]  \hspace{1cm} (55)

\[
\mathcal{W} = \begin{pmatrix}
0.0961 & -0.0052 & -0.0257 & -0.0004 \\
-0.0052 & 0.0021 & 0.0042 & -0.0001 \\
-0.0257 & 0.0042 & 0.0139 & -0.0008 \\
-0.0004 & -0.0001 & -0.0008 & 0.0003
\end{pmatrix}.
\]  \hspace{1cm} (56)

Comparing Eq. (47) with Eq. (57), it is verified that \(\bar{\theta}_1, \bar{\theta}_2 < \theta_1\) and \(\bar{\theta}_3, \bar{\theta}_4 < \theta_2\). Therefore the validity of our theory presented in Section IV is confirmed.

VI. CONCLUSION

This paper has revealed a novel property of the second-order modes of discrete-time systems under variable transformation. It has been proved that the values of the second-order modes are decreased under any variable transformation of bounded-real function. This result is the generalized version of the theory presented by Mullis and Roberts; Mullis and Roberts’ theory restricted the function for variable transformation to lossless bounded-real function, while our theory has extended the discussion on the variable transformation to the case of bounded-real function.

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